BULLETIN

OF THE

UNIVERSITY OF WISCONSIN

ENGINEERING SERIES

VOLUME VIII

1914-1917

MADISON, WISCONSIN

1917

III

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BULLETIN OF THE UNIVERSITY OF WISCONSIN

NO. 672

ENGINEERING SERIES, VOL. 8, NO. 1, PP. 1-72

THE DIAPHRAGM METHOD FOR THE MEASUREMENT OF WATER IN OPEN CHANNELS OF UNIFORM CROSS-SECTION

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MADISON, WISCONSIN
1914

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PREFACE

The diaphragm method for measuring moderately large quantities of water, when flowing in open channels of uniform cross-section, is probably new to American engineers, although the method has been in use for some time in Europe for the testing of turbines. The accuracy of the new method, however, was not determined until recently, when the Swiss Bureau of Hydrography conducted a series of experiments for that purpose. The results of these tests showed a high degree of precision, and as little, if anything, has been written on this subject in English, the following pages have been prepared with the object of bringing the device to the attention of American engineers. No experimental work with the new device has been performed as yet at the hydraulic laboratory with which the writer is connected, so that the bulletin has been prepared from the knowledge obtained by a study of the existing literature on the subject, from which the essential portions of the bulletin have been abstracted.

The writer is indebted to the following persons for kindly furnishing the photographs from which the illustrations were made: Professor Ernst Reichel, Königlichen Technischen Hochschule, Berlin, Germany; Otto Lütschg, Adjunkt der Schweizerischen Landeshydrographie, Bern, Switzerland; and J. M. Voith, of the firm of J. M. Voith Maschinenfabrik, Heidenheim, Germany. Credit is due to Messrs. Phinney, Rather, Todd and Youngberg, students in the college of engineering, for assistance in the preparation of the drawings.

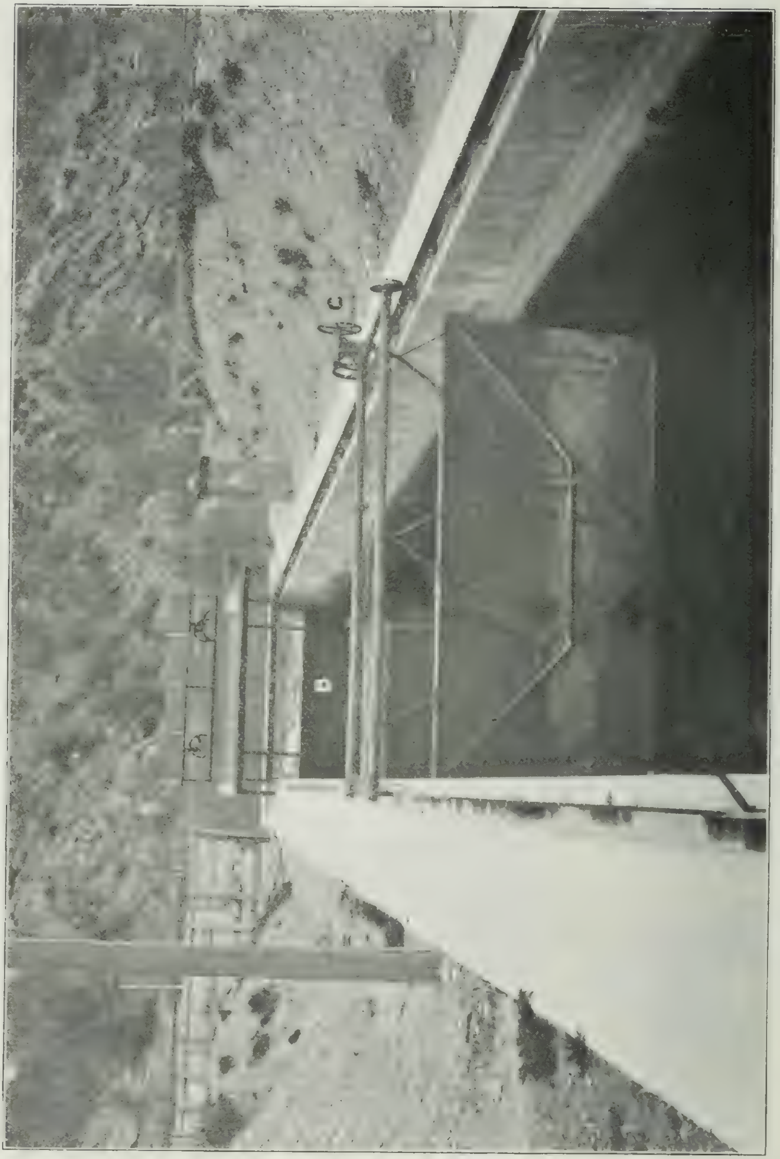


Fig. 1.—Down Stream View of Rating Canal and Diaphragm at Ackersand.

INTRODUCTION

The measurement of large quantities of water with a reasonable degree of precision is a problem, the solution of which is often a laborious, expensive, and time consuming procedure. Any device, therefore, which will simplify this operation without decreasing the accuracy of the measurement is to be welcomed. Such a device is the invention of Professor Erik Andersson of the University of Stockholm and may be termed the diaphragm method of gaging water. It was invented about nine years ago and since then the method has been used in Europe with considerable success for the testing of turbines.

Heretofore the methods available for the determination of the rate of discharge in open channels have been with current meter, float, weir, or Pitot tube. The simplest of these methods, and the only one in which a direct measurement of the velocity is obtained, is the float measurement. This method is, however, open to the objection that the floats only measure the velocity at a particular section of the stream, and a large number of observations at different sections must be made in order to obtain the mean velocity. The diaphragm method is essentially a modification of the float method, in which the velocity is integrated over the entire area, but its use is restricted to the measurement of water flowing in a channel of uniform cross-section.

The apparatus consists of a diaphragm (see Fig. 1) suspended from a car, which runs on a carefully lined and levelled track laid on the canal walls. The frame work of the car and diaphragm is built of steel tubing or light angle iron, and the wheels of the car are usually built of aluminum and run on ball bearings. The diaphragm frame is covered with oiled canvas or some light metal and swings about a horizontal axis in the direction of the current, so that it can readily be immersed

or withdrawn without causing an appreciable rise in the elevation of the water surface. When a measurement of the velocity is being made a clutch holds the diaphragm in a vertical position.

The method of procedure in obtaining a gaging is quite simple. A measured distance is first laid off along the canal walls and the diaphragm is then dropped into the stream at a point sufficiently far upstream, so that it will have obtained uniform motion by the time it reaches the beginning of the measured distance. The time of transit over this distance is then observed, from which the velocity of the diaphragm can be computed. The mean velocity of the water is then usually assumed to be the same as the velocity of the diaphragm. A correction should, however, be made for the frictional resistance of the car, and for the velocities in the clearance between the diaphragm and the periphery of the canal, which are not integrated by the diaphragm. In well designed apparatus this frictional resistance and the clearance—usually about half an inch—are so small that no serious error is caused by neglecting their effects, especially since they have a tendency to counteract each other on account of the smaller velocities occurring near the periphery. To obtain the discharge the cross-sectional area of the water must be known; the depth of water in the channel is, therefore, constantly observed during a gaging.

The chief advantage in this method is the rapidity with which the measurement can be made. This is of importance especially in the testing of turbines, where it is rather difficult to keep operating conditions constant for a time sufficient to obtain a good current meter measurement. In testing turbines at power stations, weir measurements are usually impracticable on account of sacrificing part of the available head, and current meter gagings, although the least expensive, require skilled observers, an accurate rating of the instrument, and considerable time in computing the discharge. A complete diaphragm gaging can be made in but a few minutes and the result immediately determined.

The disadvantages of the diaphragm method are that a channel of sufficient length and uniform section must be available, and that the cost of installing the necessary apparatus is rather high.

For these reasons the method is limited to the measurement of moderately large quantities of water, and its application will probably be restricted to hydraulic laboratories, turbine testing stations, and places where the apparatus can be used often and first cost is, therefore, not such an important factor. For high-head power plants or small low-head plants, the installation of the diaphragm apparatus would be of considerable commercial importance, inasmuch as the operating efficiency of the turbines could be obtained at all times with a few simple measurements. The new method could also be applied to discharge measurements in the main laterals of irrigation systems, where considerable trouble has been experienced with weir measurements on account of the silt deposits. With the diaphragm the velocities can be made high enough so that this drawback will be overcome.

Professor Andersson has used the method extensively in the Scandanavian countries for testing turbines, and in places where a suitable canal was not available a flume of wood was built. Wherever it was possible a length of from 50 to 100 feet was chosen for the rating flume, but in some cases it was only possible to get a length of 33 feet. The actual gaging distance was then only from 10 to 13 feet. Some of the canal sections were quite large, as quantities as high as 830 cubic feet per second have been measured.

When the diaphragm assumes large proportions and the length of the gaging distance is small, it becomes impractical to withdraw the diaphragm from the water after each measurement. The diaphragm is, therefore, built with openings which are covered with flaps when the measurement is taken. In some cases where the car and track were not available, the diaphragm was guided by wooden strips in the passage over the gaging distance.

It has been suggested that the diaphragm be made flexible, as shown in Figs. 2 and 3, so that it will conform to the shape of the vertical velocity curves. For this purpose the canvas is fastened only to the lower edge of the frame, the upper part being rolled up on a wooden float suspended from two cables. The sides of the canvas can be stiffened either with a strip of leather or rubber, and the roller permits adjusting the length to the depth of water. It is questionable whether there is any distinct

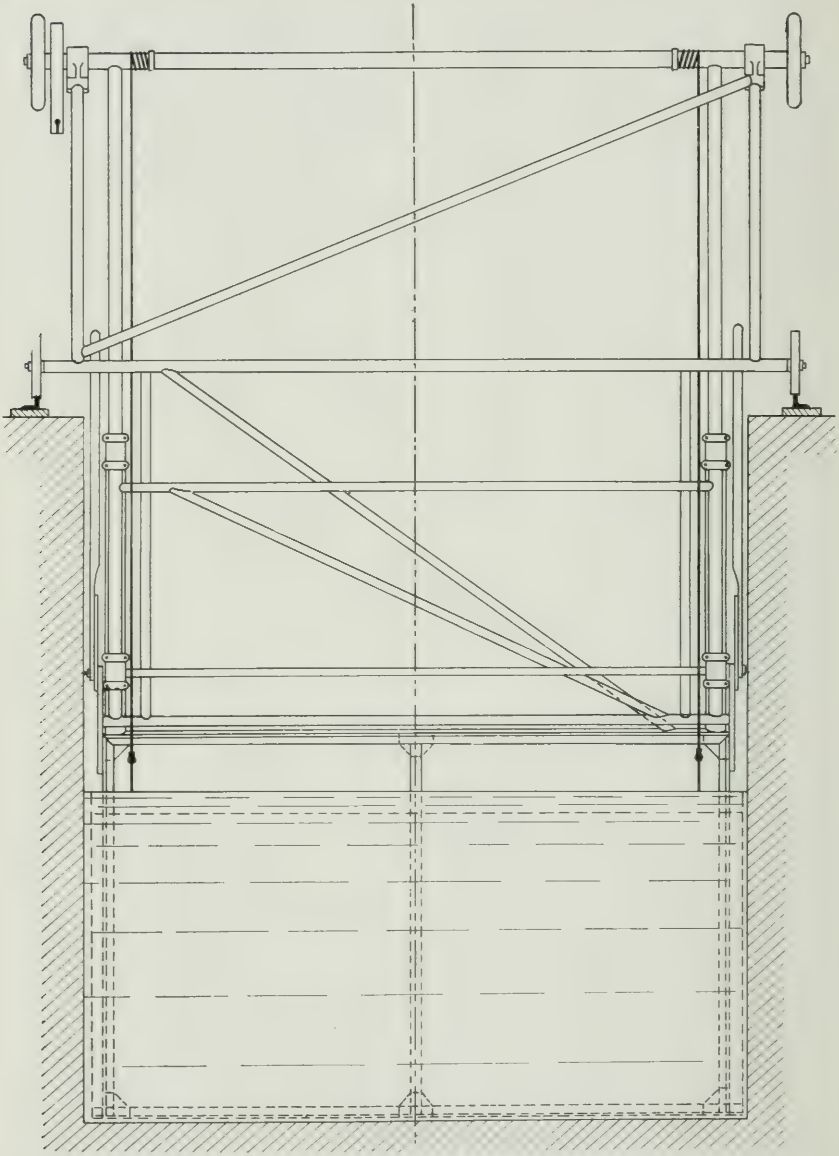


FIG. 2.—ILLUSTRATION OF DIAPHRAGM WITH FLEXIBLE COVERING.

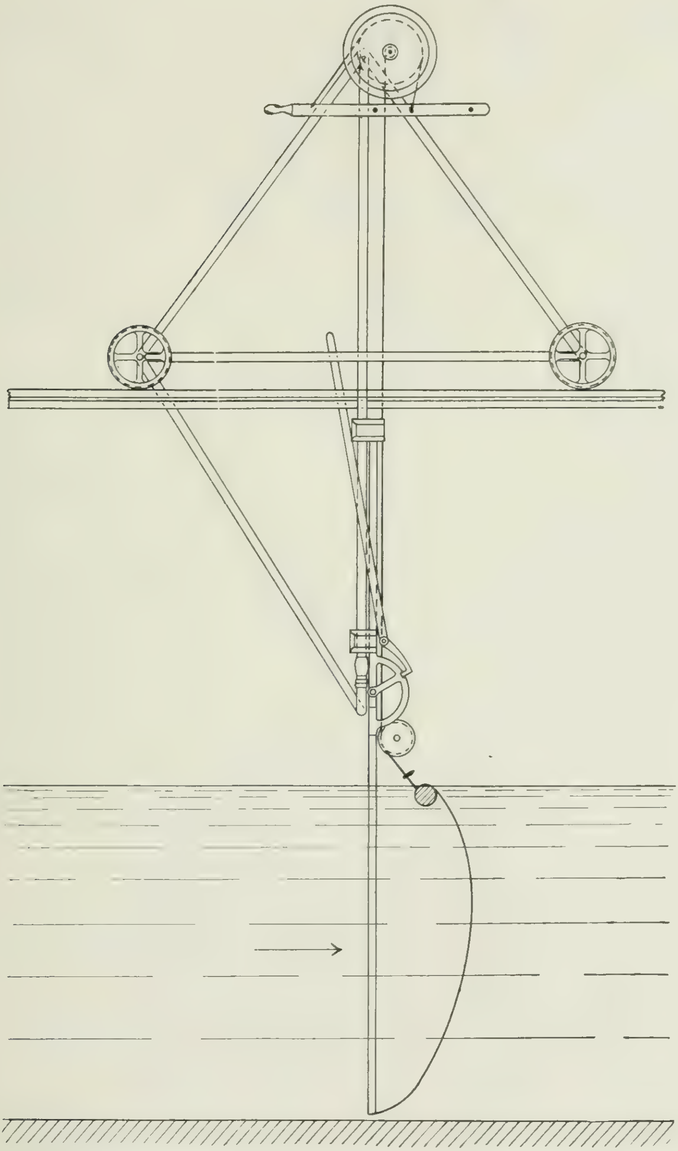


FIG. 3.—ILLUSTRATION OF DIAPHRAGM WITH FLEXIBLE COVERING.

advantage in this arrangement and its operation is more cumbersome than the vertical diaphragm.

Diaphragm measurements are subject to the following errors:

(1) in observing the time,

(2) in observing the depth of water,

(3) the frictional resistance of the car,

(4) the velocities in the clearance,

(5) the wind pressure on that part of the apparatus not submerged,

(6) the diaphragm rubbing against the periphery of the canal,

(7) foreign matter becoming wedged between the diaphragm and periphery.

If precaution be taken to avoid these errors, the diaphragm method offers an accurate means of measuring moderately large quantities of water, which can not be exceeded by any other method known at the present status of the science of hydrometry.

In the following pages various installations of the diaphragm and method of using it have been described, and the results of the experiments to determine its accuracy are given in detail.

DESCRIPTION OF THE APPARATUS AND METHOD OF MEASURING DISCHARGE AT THE J. M. VOITH TURBINE TESTING STATION

One of the earliest installations of the diaphragm was made at the J. M. Voith turbine testing station at Heidenheim, Württemberg, Germany.[1] At this station there are two canals, (see Figs. 4 and 5) each 9.84 feet wide, 4.59 feet deep and 72 feet long. The one on the right is the tail-race for the two turbines operating the plant, while the one on the left serves as a rating canal for the testing of turbines. The latter was carefully built of uniform section throughout, so that it is particularly adapted for diaphragm measurements.

Description of Car and Diaphragm.—Angle-iron rails (A, Fig. 6) with the upper edge machined were laid on each side of the canal parallel to the walls and accurately levelled. A car (see Fig. 6) built in the form of a T and consisting of a frame and three wheels runs on this track. The frame is built of steel tubing, 2 $\frac{3}{8}$ inches in diameter and $\frac{1}{16}$ of an inch thick. The entire car was made as light as possible to avoid excessive frictional resistance. In order to avoid binding when the car is in motion, only the two wheels running on the same rail are grooved, the third being flat.

The diaphragm is hung from the car and is movable about the hinges at D, Fig. 6. It is built of light angle-iron and wooden strips covered with varnished canvas. A clutch at F holds the diaphragm in a vertical position during the gaging, and at the end of the run is released automatically by striking the trip at

[1] This plant is described in a book entitled *Die Versuchs- und Prüfstationen für Wasserturbinen der Firma J. M. Voith in Heidenheim a. d. Brenz, Württemberg, und St. Pölten, Nieder-Österreich*, published in 1909 by Julius Springer, Berlin, Germany.

FIG. 4.—VIEW OF RATING CANAL AND DIAPHRAGM AT THE J. M. VOITH TURBINE TESTING STATION.

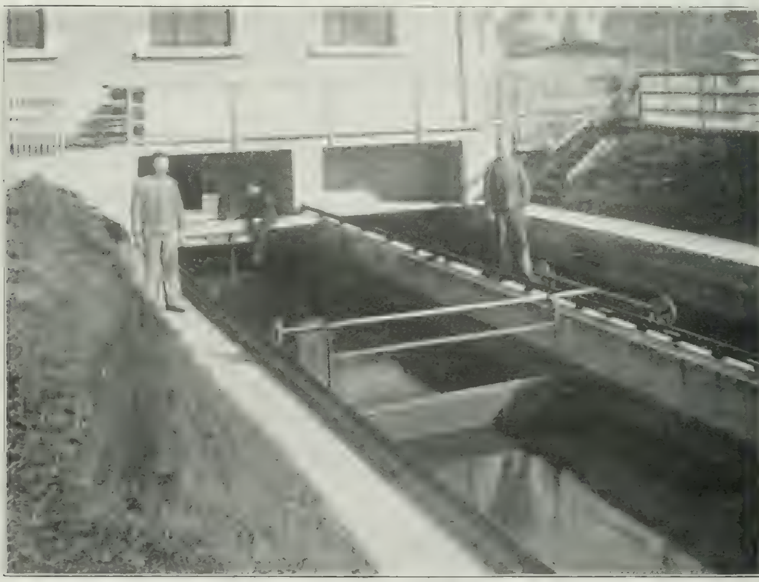


FIG. 5.—VIEW OF DIAPHRAGM DURING A GAGING AT THE J. M. VOITH TURBINE TESTING STATION.

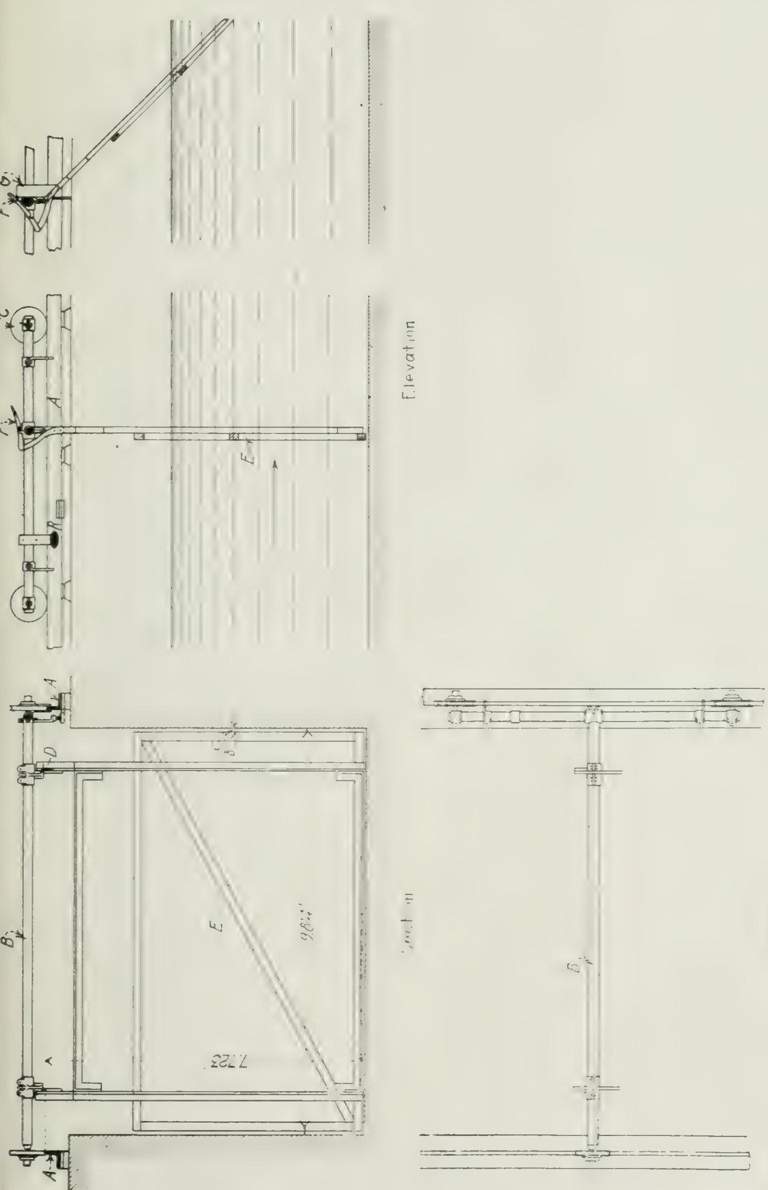


Fig. 6.—Details of the Diaphragm and Car at the J. M. Voith Turbine Testing Station.

G. The car and diaphragm weigh 88 pounds and a force of 0.9 of a pound is capable of moving it along the track when the canal is empty.

Method of Making a Gaging.—In making a gaging two men first hold the diaphragm out of the water with ropes, as shown in position I, Fig. 7. At a given signal it is dropped quietly into the water and the current carries the car and diaphragm along. The diaphragm dips into the water quietly without causing any commotion or formation of wave action, so that the regimen of the stream is not disturbed. After a short distance the diaphragm hangs in a vertical position, as shown in position II, and is held there by the clutch F. The actual length of the gaging distance is only 32.8 feet, as the diaphragm must travel over a considerable distance before the velocity becomes uniform. The time of transit is taken with a stop-watch and the velocity can then be computed immediately. As soon as the clutch strikes the trip at G, the diaphragm is released and takes the position shown as III.

During the run observers read the depth X of the water surface below the top of the rails at points H and I, Fig. 7. The readings at these points as a rule do not vary more than 0.005 of a foot; the depth of water in the channel is, therefore, computed from the mean of these readings. The average depth of the bottom of the canal from the top of the rail is 7.723 feet and the average width is 9.816 feet. The discharge is then computed from

$$Q = (7.723 - X) \times 9.816 \times v$$

in which v is the mean velocity of the diaphragm.

Accuracy.—The clearance between the diaphragm and periphery of the canal is about 0.4 of an inch. As the smaller velocities occur in this part of the cross-section, it would seem that the velocity as indicated by the diaphragm would be larger than the actual mean velocity of the water. It seems, however, that this error is offset by the frictional resistance of the car and no correction is made for it. A few experiments made at this station with calibrated orifices to check the diaphragm method showed a close agreement, and that the diaphragm gave accurate measurements with velocities as low as 0.02 of a foot per second.

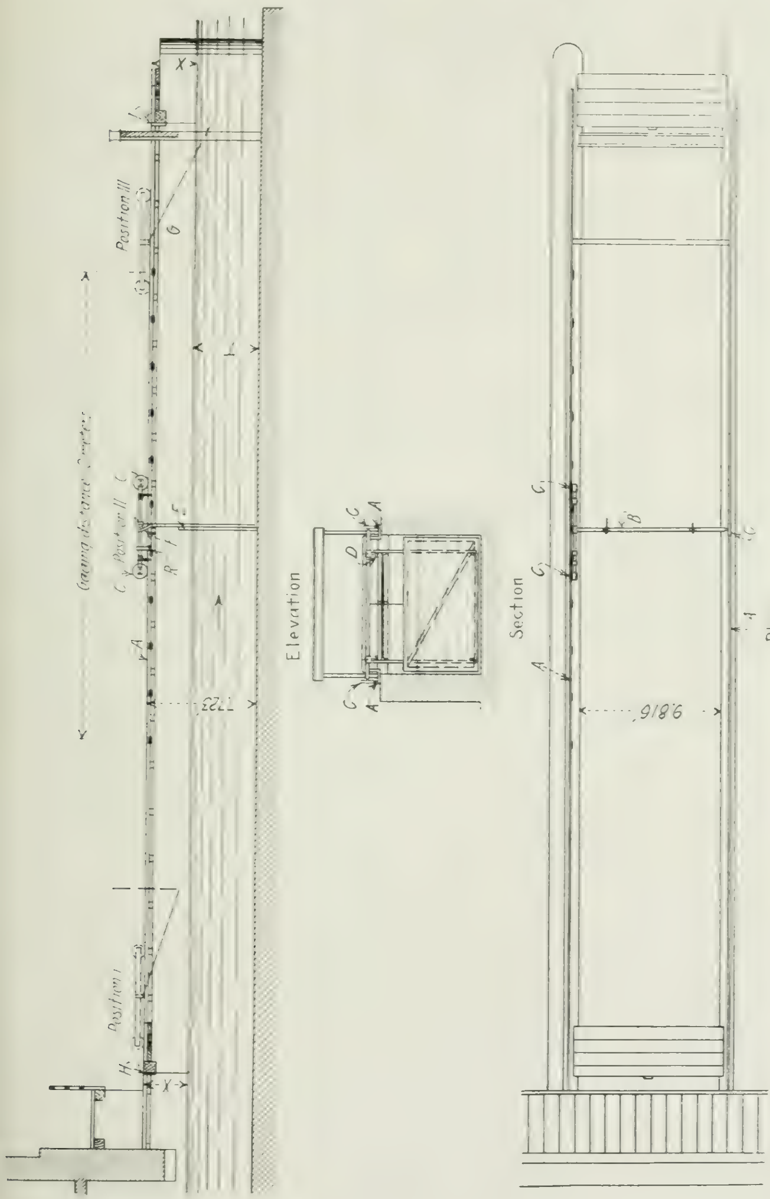


FIG. 7.—PLAN AND ELEVATION OF RATING CANAL AT THE J. M. VOITH TURBINE TESTING STATION.

The largest quantity of water measured at this station is approximately 141 cubic feet per second, corresponding to a velocity of 2.79 feet per second. The time of transit with this velocity is about 12 seconds. It was found that errors in the time measurement amounting to $\pm \frac{1}{4}$ to $\frac{2}{5}$ of a second were possible, when the time was observed with a stop-watch. With the largest discharge this error amounts to about 2.5 per cent but is correspondingly smaller with the smaller discharges.

Description of Recording Device.—In order to gain more accuracy in observing the time and also to obtain simultaneous

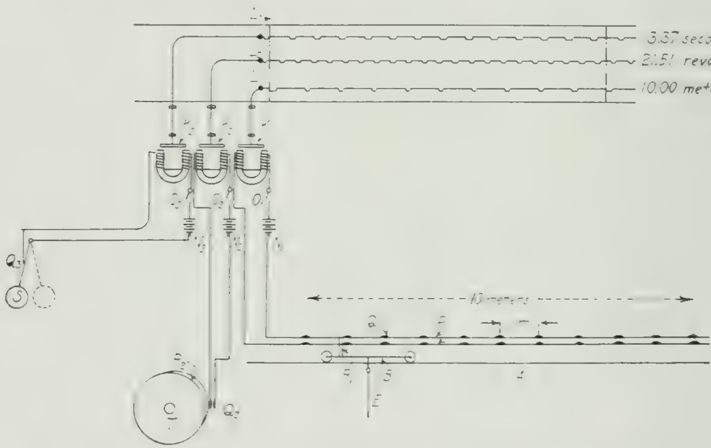


FIG. 8.—SCHEMATIC ARRANGEMENT OF RECORDING APPARATUS AT THE J. M. VOITH TURBINE TESTING STATION.

readings of the speed of the turbine during a gaging, an electrical recording device was devised, the schematic arrangement of which is shown in Fig. 8. K_1, K_2, K_3 are three electromagnets; pens L_1, L_2, L_3 are attached to the armatures of these in such a way that a break in the line, traced on a moving strip of paper, occurs whenever the circuit is closed. The paper is actuated by clock work. Next to the rail A, an electric circuit P was arranged with 11 contact points spaced 3.28 feet apart. The battery N_1 and switch O_1 are connected in series with the electromagnet K_1. Then, when the shoe R_1 fastened to the car B passes

FIG. 9.—REPRODUCTION OF RECORDING CHART SHOWING FOUR RUNS MADE AT THE J. M. VOITH TURBINE TESTING STATION.

over the contact points Q_1, the circuit is closed and this is shown on the paper by a break in the line traced by the pen L_1. A similar arrangement is used to record the number of revolutions of the turbine. The eccentric R_2 on the turbine shaft T closes the circuit at Q_2, and the pen L_2 records a break in the line for each revolution. The third pen L_3 is connected to the time circuit, every ½ second being recorded on the chart.

The method of procedure in making a turbine test is as follows: sufficient time is allowed for the discharge to become steady, after which the brake is adjusted. At a given signal the

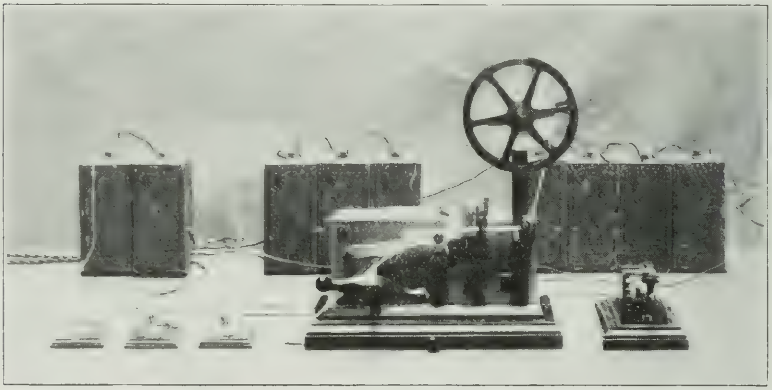


FIG. 10.—VIEW OF RECORDING APPARATUS USED AT THE J. M. VOITH TURBINE TESTING STATION.

diaphragm is then dropped into the water, the switches O_1, O_2, O_3 on the electric circuits are closed, and the recording drum is set in motion. The time, number of revolutions of the turbine, and the velocity of the diaphragm are then recorded automatically upon the chart, but the elevation of the water surface in the canal must be observed at gages H and I, Fig. 7. Vertical lines are then drawn on the chart at the beginning and ending of the gaging distance, which cut off the corresponding time and number of revolutions. Thus, for the diagram shown in Fig. 8, the time in traveling over the gaging distance of 32.8 feet was 13.37 seconds, and the number of revolutions of the turbine during that time was 21.51. Hence,

$$n = \frac{21.51}{13.37} \times 60 = 96.5 \text{ R. P. M.},$$

$$v = \frac{32.8}{13.37} = 2.45 \text{ feet per second.}$$

The recording chart moves at a sufficiently high rate, in this case 0.033 feet per second, so that an error in scaling the chart is inappreciable. A reproduction of the recording chart showing four runs is shown in Fig. 9, and a view of the recording apparatus in Fig. 10.

The discharges at this station were formerly measured by current meters, as the head was too small to permit using a weir. A current meter gaging at 8 points in the cross-section took from 1/2 to 3/4 of an hour, during which time it was difficult to keep operating conditions constant. The time for computing the result is also an important factor, and if one considers that several measurements of the discharge must be made for a complete turbine test, it may readily be seen what a saving in time the diaphragm method accomplishes. A complete test of a turbine can be made in half a day with the new method, and the results are considered quite as accurate as with any of the usual methods employed in the testing of turbines.

DESCRIPTION OF THE DIAPHRAGM APPARATUS AT THE BERLIN TECHNISCHE HOCHSCHULE

In the testing of turbines at the hydraulic laboratory of the Technische Hochschule at Berlin, the diaphragm method of measuring discharge has now been used for a number of years. The facilities for the installation of this method were not very favorable at this place, as the tail-race canal (see Fig. 11) is built of brick and is only 33 feet long, but nevertheless an instrument was built which gives satisfactory service and accurate measurements.

Description of Car and Diaphragm.—The structural details of the diaphragm (see Figs. 12 and 13) and recording device differ somewhat from those designed at the Voith testing station. The frame work of the car is made of very thin steel tubing, brazed together and although weighing but 88 pounds is very rigid. The diaphragm is built of light angle-iron covered with oiled canvas and is hung from the horizontal axis A, Fig. 12, about which it is free to swing in a forward direction. It can also be raised or lowered by two small cables attached to the two hand wheels N, the guides K sliding along the two vertical tubes T. A brake B, Fig. 13, is attached to one of the wheels so that the speed with which the diaphragm is lowered can be easily regulated. Its descent is limited by the two rubber buffers P. When in a vertical position the diaphragm is held rigid with the vertical frame by means of the clutch R, and when the clutch is released the current swings it around the axis A to the dotted position shown in Fig. 13.

The bottom and sides of the canal were carefully plastered with cement mortar, so that the clearance between the diaphragm and walls is only about 0.2 to 0.3 of an inch. A distance of

about 10 feet is necessary for immersing the diaphragm, so that the actual length of the gaging distance is only 23 feet. Experience has shown, however, that this distance is sufficient with the rapid vertical immersion of the diaphragm.

Method of Procedure.—The method of procedure in making a discharge measurement is as follows: the car is placed at the upstream end of the canal, with the diaphragm raised but locked in a vertical position with the sliding frame. At a given signal

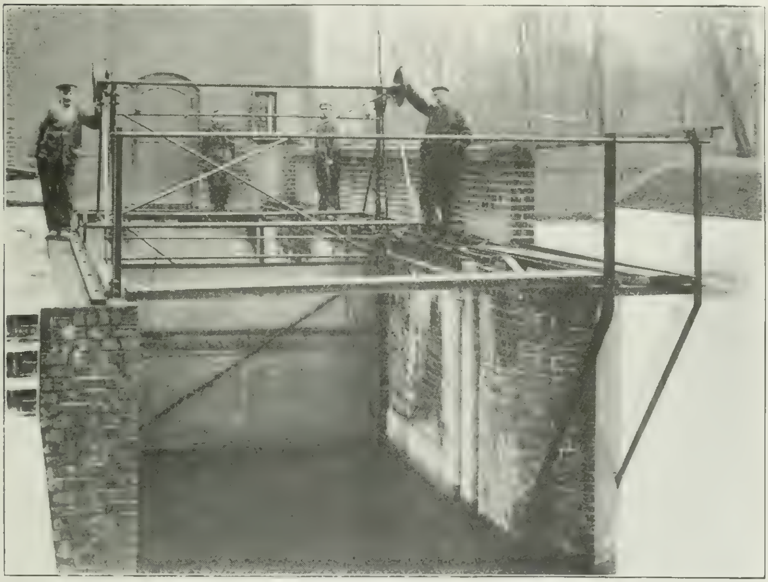


FIG. 11.—VIEW OF RATING CANAL AND DIAPHRAGM AT THE BERLIN TECHNISCHE HOCHSCHULE.

the diaphragm is dropped, its descent being controlled by the hand brake, so that it falls gently against the rubber buffers. These buffers are set so that just a small clearance occurs between the diaphragm and the bottom of the canal. As soon as the diaphragm is partly immersed the car begins to move, but the diaphragm usually reaches its lowest position before the car has traversed 3 or 4 feet. After a distance of 10 feet the motion of the car is so uniform that the remaining distance can be used for determining its velocity. As soon as the car reaches the end

of the canal it is held by two bumpers which also release the clutch; the diaphragm then swings about the axis A, the frame is raised and the car is pulled back to the starting point. As the entire operation can be completed in but a few seconds, several

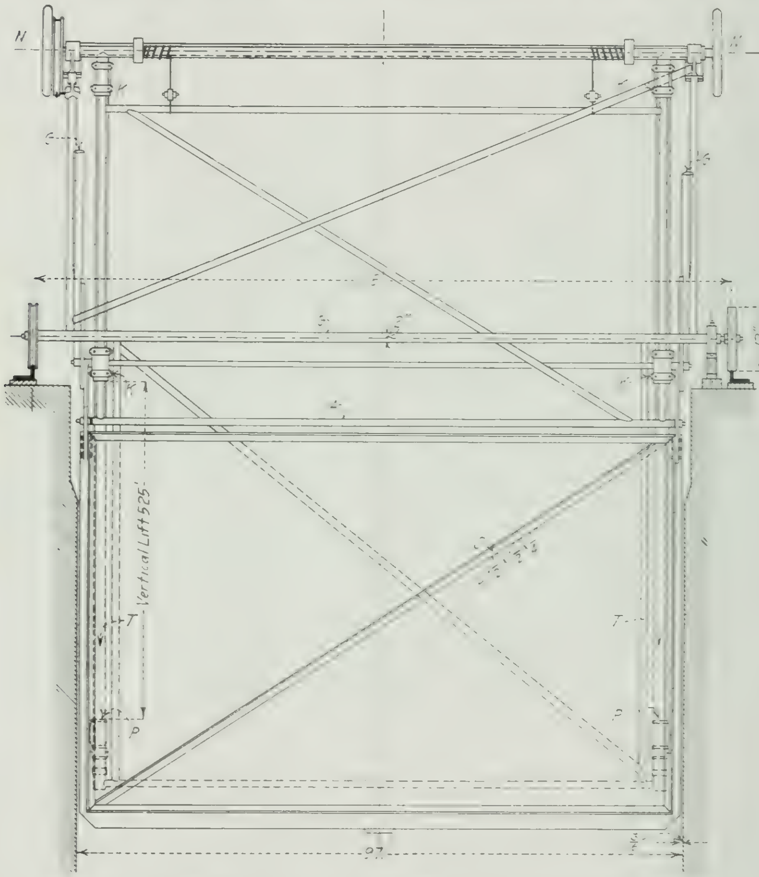


FIG. 12.—DETAILS OF THE DIAPHRAGM AND CAR AT THE BERLIN TECHNISCHE HOCHSCHULE.

measurements can be made as a check while the turbine is held under constant conditions. The depth of water in the gaging channel is read several times during a run by means of a float-gage located in a recess of the canal. The back water occasioned

by the immersion of the diaphragm is insignificant and does not affect the accuracy of the measurement, the difference between

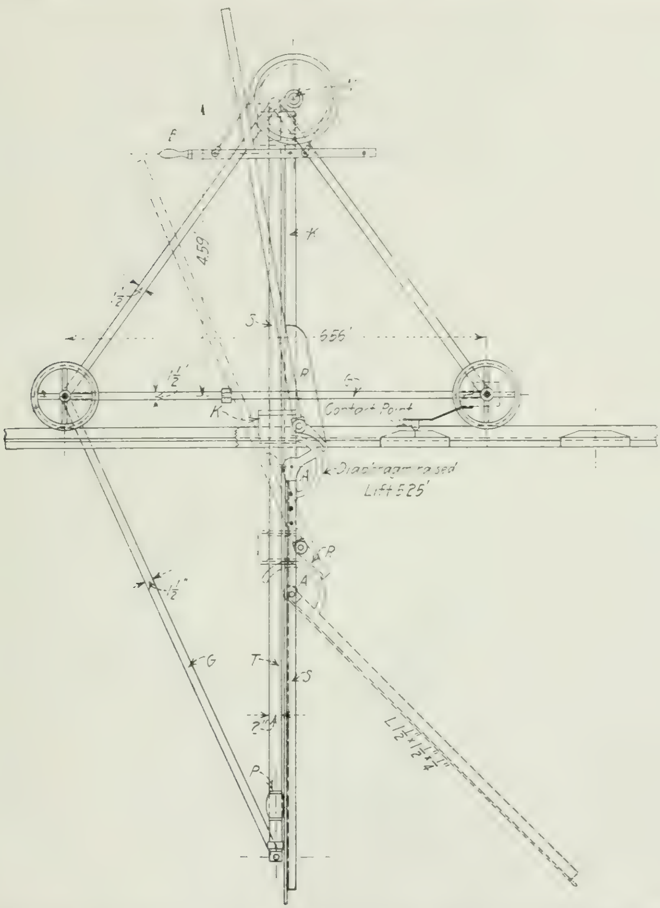


FIG. 13.—DETAILS OF THE DIAPHRAGM AND CAR AT THE BERLIN TECHNISCHE HOCHSCHULE.

the levels of the water on the upstream and downstream sides of the diaphragm being scarcely 0.007 of a foot.

Description of Recording Device.—As a great many turbines are tested at the laboratory of the Berlin Technische Hochshule during the course of a year, it was necessary to devise some

scheme whereby these tests could be made with a minimum amount of time and labor and still maintain a high degree of

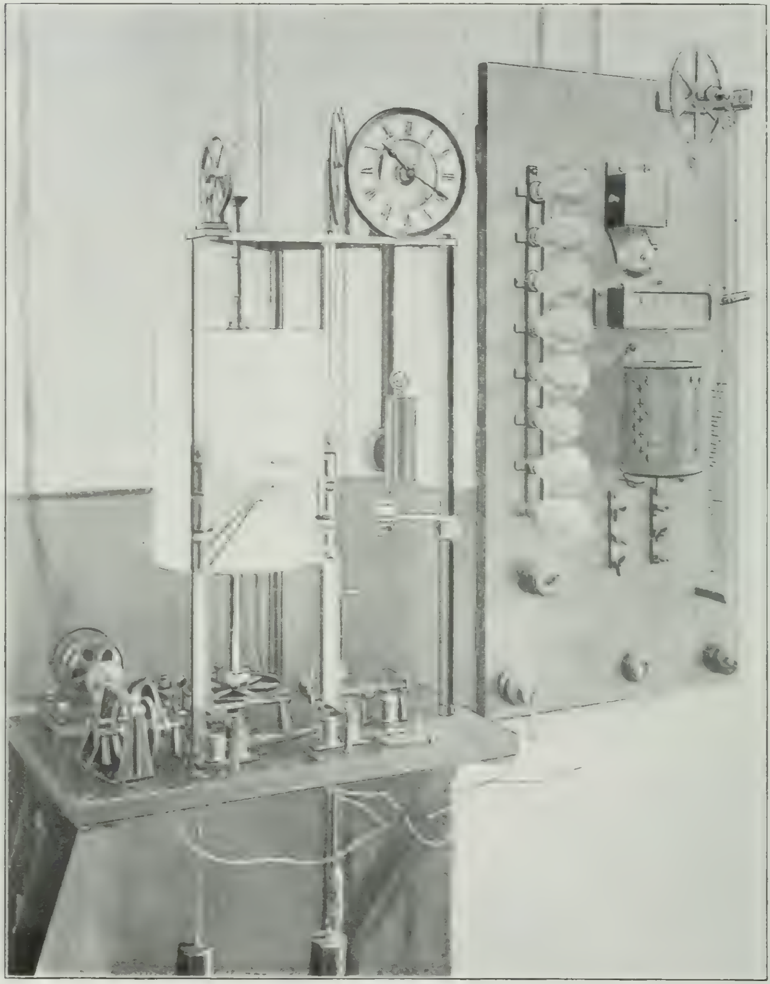


FIG. 14.—VIEW OF RECORDING APPARATUS USED AT THE BERLIN TECHNISCHE HOCHSCHULE.

accuracy in the observed measurements. The automatic recording device, shown in Figs. 14 and 15, fulfills the above require-

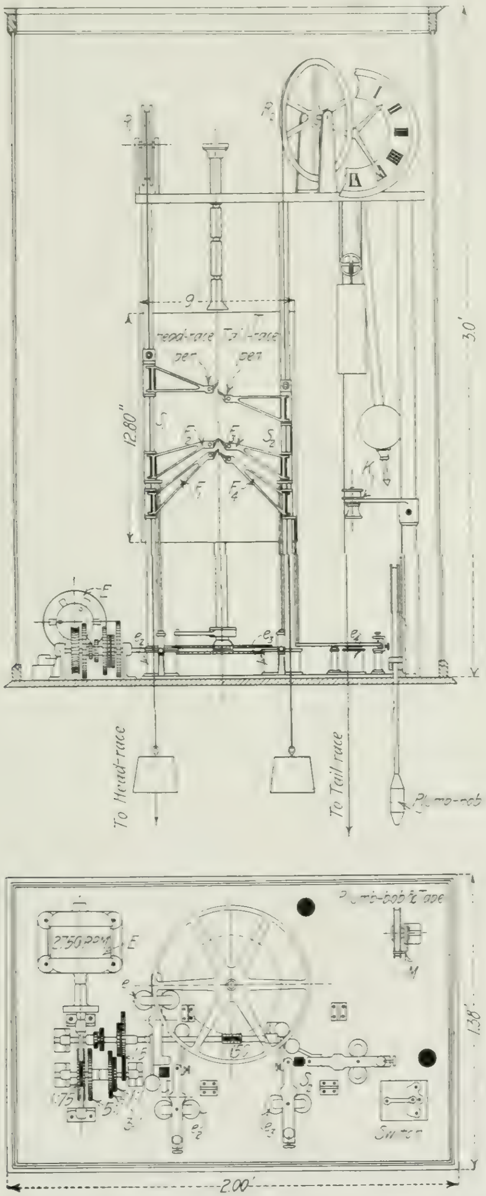


FIG. 15.—DETAILS OF RECORDING APPARATUS USED AT THE BERLIN TECHNISCHE HOCHSCHULE.

ments admirably, and was perfected by Professor Ernst Reichel. It registers the number of revolutions of the turbine, the water levels in the head- and tail-races, the velocity of the diaphragm, and the time. It consists of a drum T, Fig. 15, 9 inches in diameter and 12.8 inches long, revolving on a vertical axis. The drum is driven by a small electric motor E and the gearing is so arranged that four different speeds can be obtained, ranging from $^1/_4$ to $^5/_8$ of an inch per second. The motor is rated at $^1/_{40}$ horsepower and has a normal speed of 2750 R. P. M. The drum can be easily removed from its bearings and covered with paper; four different tests can, however, be recorded on the same piece of paper, as the drum is movable along the vertical axis. The entire mechanism is mounted on a heavy cast-iron base plate, and there are four vertical columns capped with a plate upon which a pendulum clock and two sheaves are mounted.

A float-gage is set in a recess of the head-race and another in the tail-race. Small bronze wires attached to these gages are conducted over the sheaves R_1 and R_2 to the recording apparatus. Weights are attached to the ends of the wires to keep them taut. Columns S_1 and S_2 serve as guides for recording pens. The upper pen of column S_1 is clamped to the wire attached to the head-race float-gage, and the upper pen of column S_2 to the tail-race gage. Above each float-gage a bench mark M, Fig. 16, has been established, from which the distance T to the upper edge of the float can be read with a steel tape and plumb-bob. With the float in this position a datum line, Fig. 17, is traced on the recording chart, the absolute elevation of which is then computed by subtracting the distance T + t from the elevation of the bench mark, t being a constant easily determined from the depth of flotation. The actual variations in the water levels in the head- and tail-races are recorded on the chart (see Fig. 17), from which the total head can then be scaled. If the precaution be taken to see that the floats are clean and that the sheaves turn easily, an accurate continuous graph of the variation in the head is obtained, errors of observation are eliminated and it is possible to dispense with the services of two observers.

The remaining pens are operated by levers attached to the armatures of small electro-magnets, so that when a circuit is

closed the pen is moved a short vertical distance and a break is shown in the line traced on the chart, Fig. 17. The magnets were operated with current obtained from the city distribution system, as dry batteries were not reliable. An electric circuit,

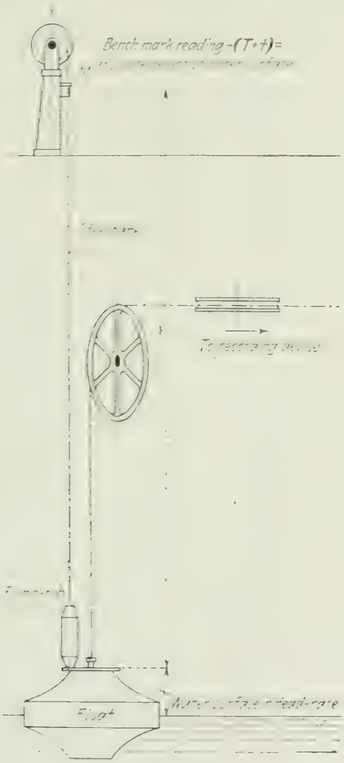
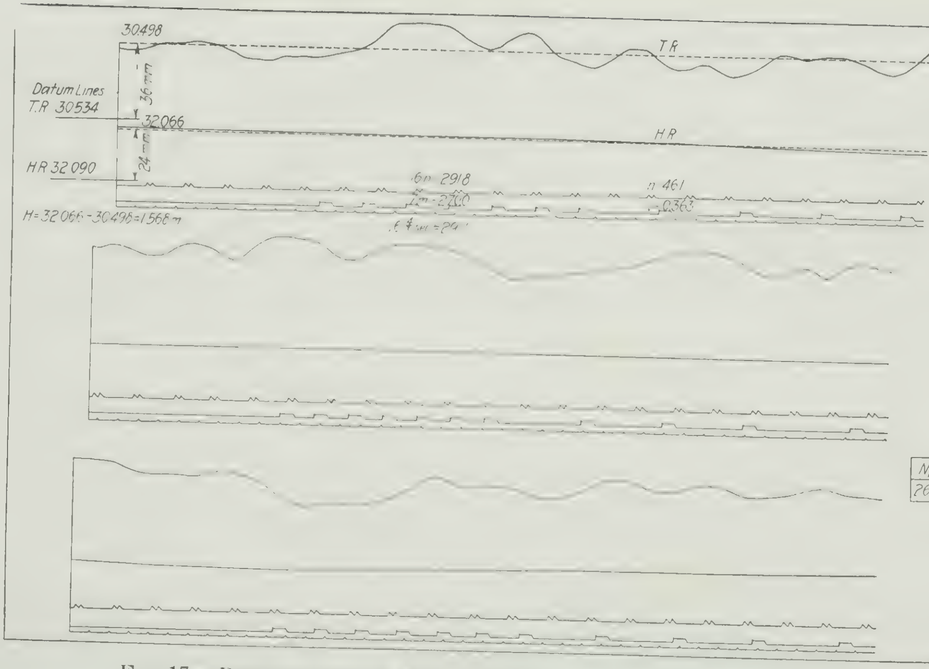


FIG. 16.—FLOAT-GAGE IN HEAD-RACE AT THE BERLIN TECHNISCHE HOCHSCHULE.

similar to the one described in connection with the apparatus used at the Voith testing station, is laid along one of the rails on which the diaphragm car runs, the contact points being spaced a half meter for part of the measuring distance and a meter apart for the rest of the distance. A graph of the transit of the diaphragm is then obtained, while two other pens operated by electric circuits record the revolutions of the turbine



Lorenz Turbine

No	Time	h	T	F	v
40	11 03	505			0363
41	11 06	506			0360
42	11 08	508			0365
Average		506.3	1159	3434	0363

No.	Q	H	P	n	Q_1
40		1.568	88	46.1	
41		1.570	88	46.6	
42		1.569	88	44.4	
Aver.	1.246	1.569	88	45.7	0.996

N_i	N_o	P_1	n_1	N_{01}	N_{i1}	Eff
26.10	16.98	56.1	36.5	8.64	13.28	651

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FIG. 17.—REPRODUCTION OF RECORDING CHART SHOWING RESULTS OF THREE TURBINE TESTS MADE AT THE BERLIN TECHNISCHE HOCHSCHULE.

h = height of water surface in m. m.; T = depth of submergence in m.; V = velocity in m. per sec.; Q = discharge in cu. m. per sec.; H = head in m.; P = brake load in kg.; n = R. P. M.; N_i = input; N_o = output; Eff. = efficiency. Subscript 1 denotes under 1 meter head.

and the time. A sixth pen is sometimes used to record the revolutions of a current meter when the discharge is being measured in this manner. The pendulum of the clock makes a contact every $\frac{2}{3}$ of a second, so that the number of revolutions of the turbine and the velocity of the diaphragm can be quickly determined from the chart. A reproduction of the results of 3 different runs taken on the same piece of paper is shown in Fig. 17. Here again a picture is obtained as to the constancy of conditions and only one observer is necessary for 5 measurements. One can also compute the result of a run in just a short time, and if consistent with the other runs it can be repeated without having the conditions of operation change appreciably. With current meter measurements it usually takes several hours to compute the discharge.

Accuracy.—With the normal depth of water in the tail-race the velocity is about 2.3 feet per second and the discharge can be obtained with great accuracy. When the velocity of the water becomes as small as 0.3 of a foot per second the measurement becomes unreliable, as each little wave, due to disturbances outside of the tail-race, then affects the motion of the diaphragm. With such small velocities, however, current meter measurements are also unreliable and in the opinion of the director of the laboratory diaphragm measurements are preferable in all cases.

A comparison between the results obtained with the current meter and with the diaphragm at this laboratory, showed a satisfactory agreement. In one of the tests the mean velocity obtained with the current meter held at 21 points in the cross-section was 2.729 feet per second, while with the diaphragm the velocity recorded was 2.709 feet per second, or a difference of about 0.7 of 1 per cent.

METHOD OF PROCEDURE IN TESTING THE TURBINES AT THE HYDRO-ELECTRIC POWER PLANT AT NOTODDEN, NORWAY

As an illustration of the application of the diaphragm method of measuring the discharge at water-power plants, where the conditions for accurate gagings are not usually as favorable as in laboratories, the following brief description of a turbine test at a high head plant in Norway may be of interest.

Method of Measuring the Discharge.—There are four units at this plant, each rated at 10,000 H. P. capacity under a head of 150 feet. With an efficiency of 80 per cent, the corresponding discharge from each unit is approximately 750 cubic feet per second. The details of construction of the diaphragm used to measure this discharge are shown in Figs. 18 and 19. The general design is similar to the ones previously described, the clearance between it and the flume being 3/8 of an inch. A recording device operated by an electric circuit placed along the flume was used to measure the velocity.

Water is supplied to this station from a storage reservoir through a canal about 490 feet long, thence through a tunnel 1670 feet long to a distributing basin, from which separate tunnels for each unit lead to the power-house situated at the bottom of a gorge. The only place available for using the diaphragm was in the intake canal leading from the storage reservoir to the tunnel, but as this canal (Fig. 20) was unlined it was necessary to build a rating flume of wood. A photographic view of this flume is shown in Fig. 20, and the details in Fig. 21. As may be seen from these figures, the flume was built the full width of the canal, as close to the walls as possible, but its depth

was only about one-half of the depth of the canal. The gaging section is 17.1 feet wide. 13.1 feet deep. and 32.8 feet long.

Considerable difficulty was experienced in getting the space between the flume and the profile of the canal water tight. and as it seemed to be impracticable to stop the flow entirely. the leakage was measured and a correction made for it. A further correction to the gross discharge as measured by the diaphragm had to be made for the loss in transmission through the tunnels. This loss was measured with the diaphragm after closing the gates to the turbines. the entire plant being shut down. The loss was found to be comparatively small. The exciter turbine as well as the main unit was fed by the same penstock and it was. therefore. necessary to determine the discharge to the former. This was obtained from a discharge curve of a turbine similarly constructed and tested at a testing station. The net discharge to a main unit was. then. the measured discharge as determined with the diaphragm. plus the leakage around the flume. minus the loss in transmission and the discharge to the exciter unit.

The head at the turbine was determined with a calibrated manometer and the total head by means of gage-floats in the head- and tail-races. The output was measured with carefully calibrated electrical instruments. the efficiency of the generator having been previously determined at the shops of the manufacturer.

Results of the Tests.—The tests were made under the direction of Professor Ernst Reichel of the Berlin Technische Hochschule. and were confined chiefly to checking the contract requirements. With a normal head of 152 feet and 250 R. P. M.. an efficiency of 78 per cent at full gate and 81 per cent at $\frac{3}{4}$ gate was guaranteed by the manufacturers. while under high water conditions. with a head of about 131 feet. the efficiency was to be 76 per cent at full gate. The results of the tests showed that the turbines exceeded the contract requirements under all conditions. With the normal head and 250 R. P. M. the measured electrical horse power was 11.750. while the contract requirement was only 10.000. With a load of 7500 electrical horse power the efficiency was 84.9 per cent and with about

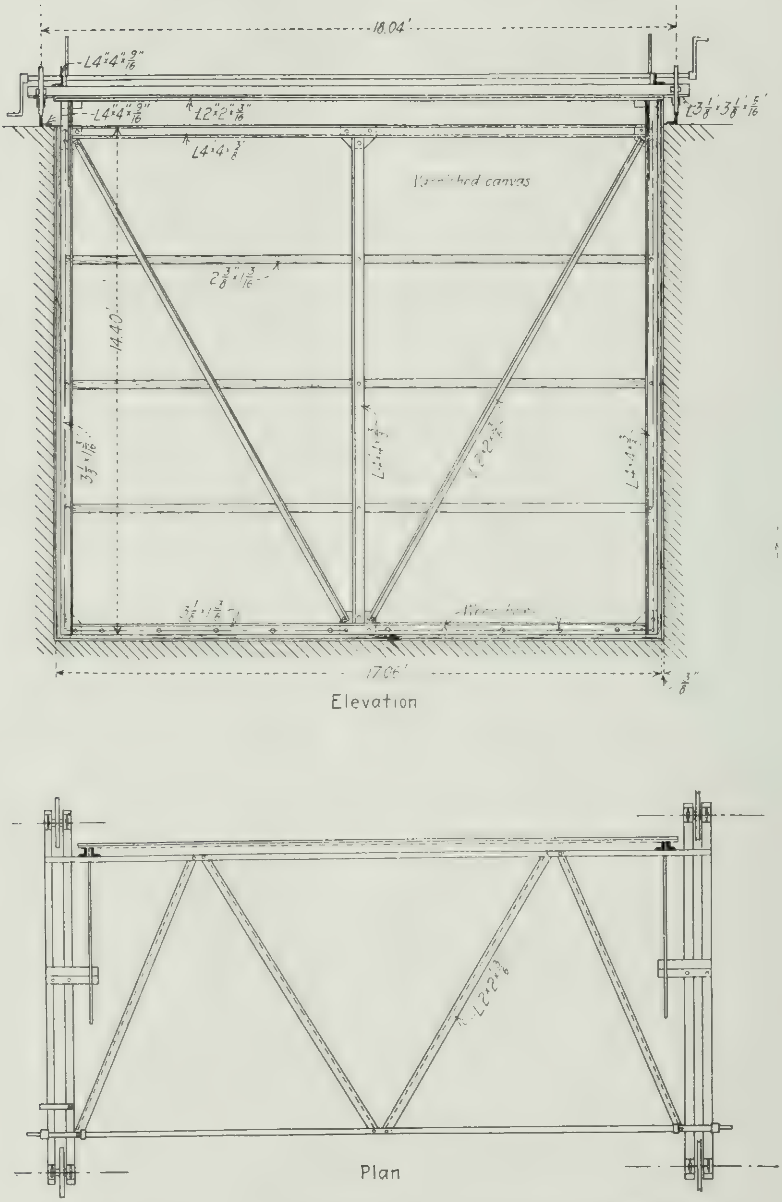

FIG. 18.—DETAILS OF DIAPHRAGM AND CAR USED AT NOTODDEN.

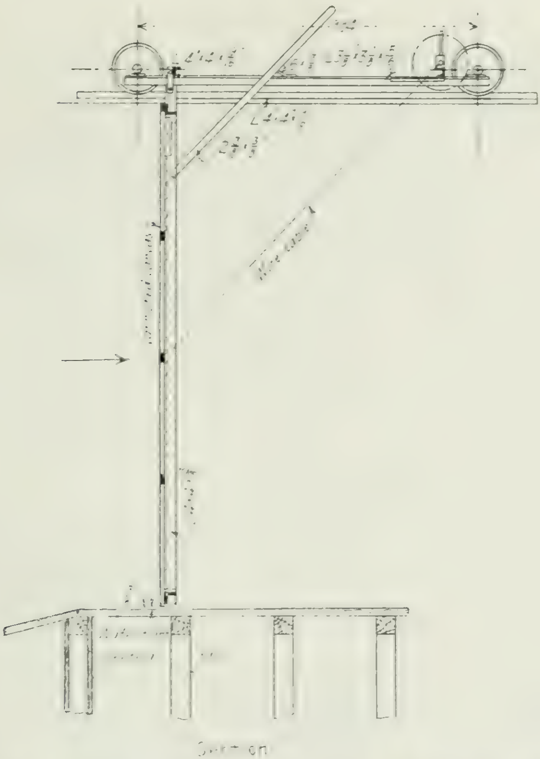

FIG. 19.—DETAILS OF DIAPHRAGM AND CAR USED AT NOTODDEN.

10,000 horse power an efficiency of 86.2 per cent was obtained. With the minimum head the power output was 9400 electrical horse power with an efficiency of 80.6 per cent, while the contract requirement was only 7650 electrical horse power with an efficiency of 76 per cent.

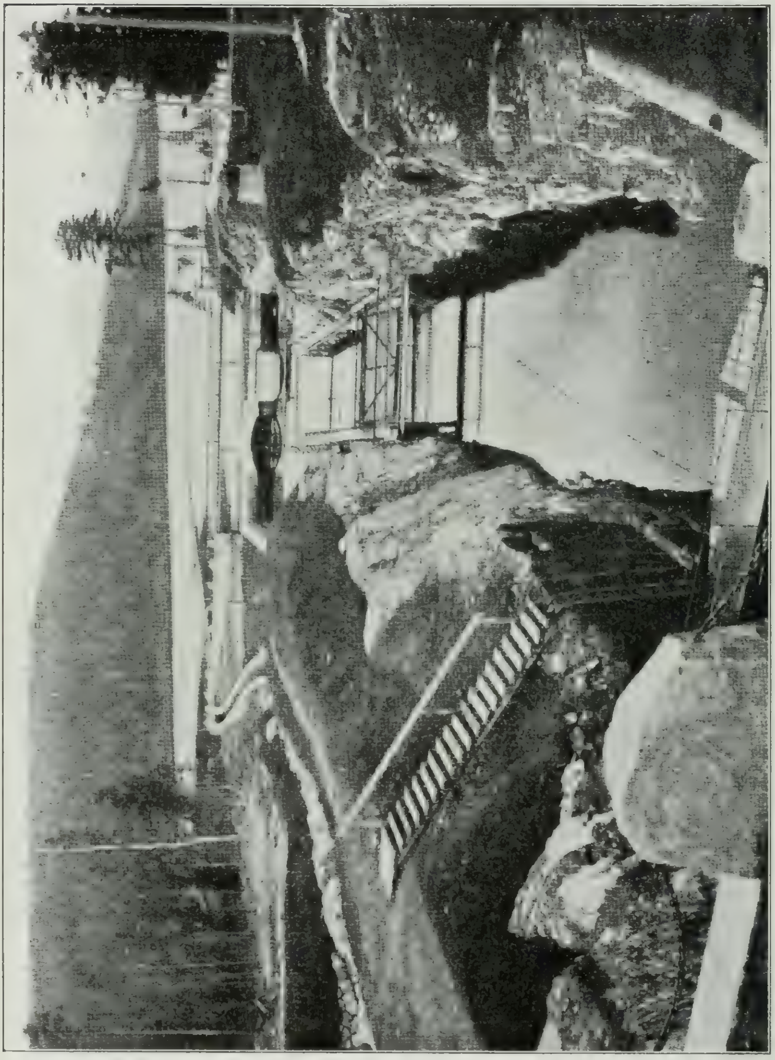

Fig. 20.—View of Rating Flume and Diaphragm at Notodden.

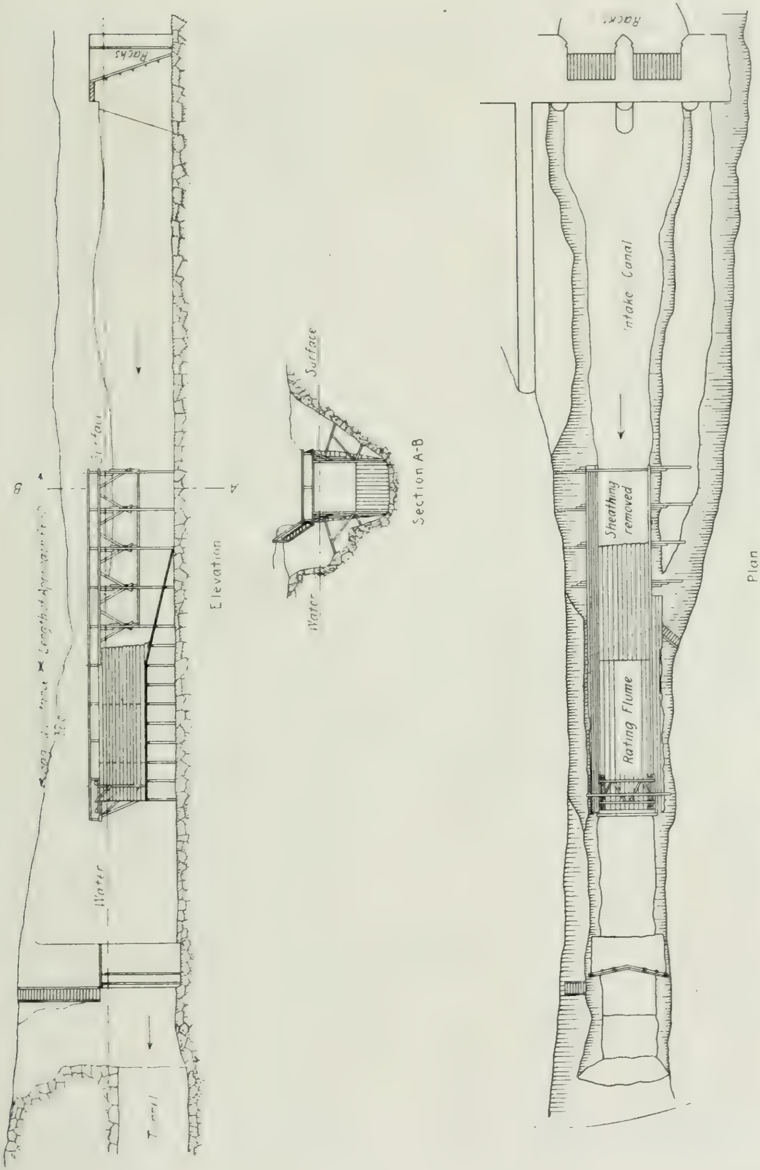

FIG. 21.—PLAN AND ELEVATION OF RATING FLUME AT NOTODDEN.

SWISS BUREAU OF HYDROGRAPHY EXPERIMENTS TO DETERMINE THE ACCURACY OF DIAPHRAGM GAGINGS.

An extensive series of experiments was made in 1912 by the Swiss Bureau of Hydrography, to determine the accuracy of diaphragm measurements. These experiments were performed at the Lonza hydro-electric plant at Ackersand, Switzerland, by Otto Lütschg and the results have been reported by him in a bulletin[2] published by the Bureau. The information presented in the following pages in regard to these experiments has been abstracted from that bulletin. The Lonzo plant[3] consists of 5 units of the Pelton type, each rated at 5500 horsepower and operated under a head of 2300 feet. Provision for using the diaphragm was made in the construction of the plant by building the tail-race of uniform section and of considerable length. The diaphragm and its appurtenances were installed by the J. M. Voith Co. of Heidenheim, Germany, at a total cost of about $580, including the recording apparatus.

The Gaging Canal.—The gaging canal (see Figs. 22, 23 and 28) is of rectangular section, 11.4 feet wide, 6.6 feet deep and 98.4 feet long. It is built of uniform section throughout its length. At the down-stream end a vertical sliding gate B, Fig. 22, regulates the depth of water in the canal. When closed this gate also serves as a weir. At the upstream end a vertical sliding baffle D, Fig. 23, was built to eliminate turbulent flow. The length of the gaging distance is 46 feet.

[2] *Vergleichs-Versuche mit Flügel- und Schirm-Apparat zur Bestimmung von Wassermengen*,—by Otto Lütschg, Adjunkt der Schweizerischen Landeshydrographie. Bern, Switzerland: The Secretary, der Schweizerischen Landeshydrographie. 1.50 Francs.

[3] This plant is described in an article published in *Schweizerische Bauzeitung*, Vol. 54, Nov. 6, 1909, page 263.

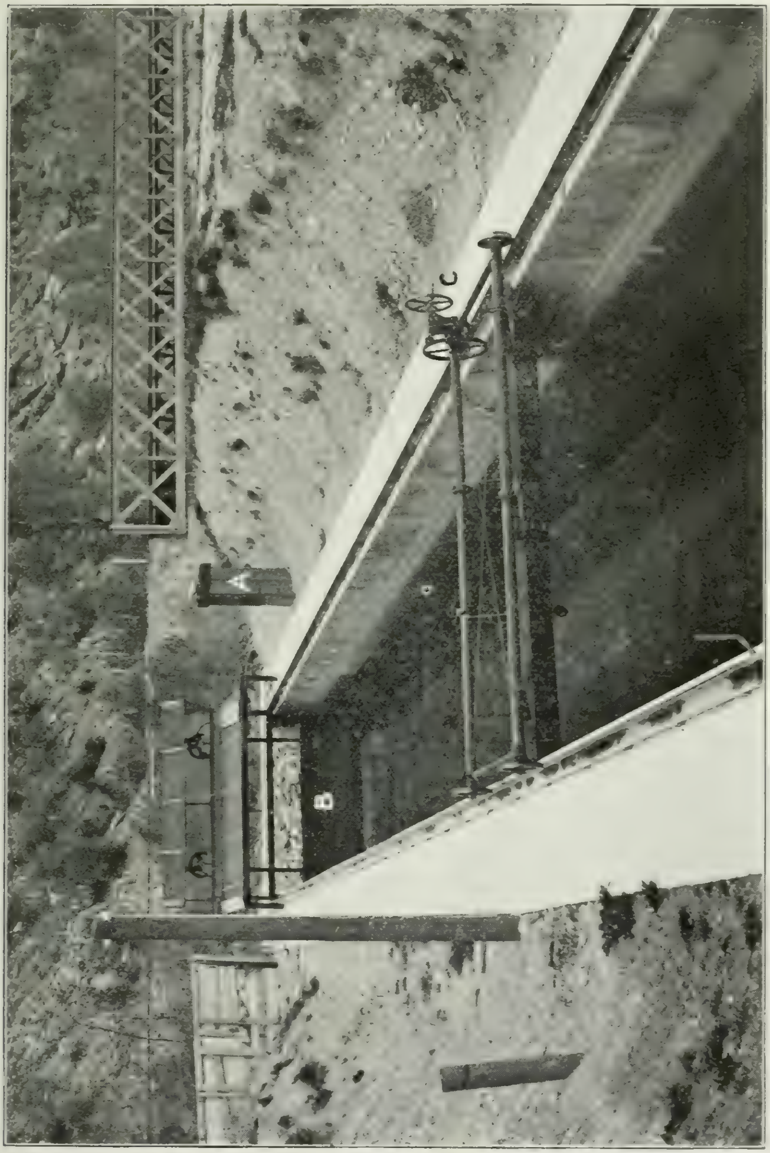

FIG. 22.—DOWN STREAM VIEW OF RATING CANAL AND DIAPHRAGM AT ACKERSAND.

The Diaphragm.—The structural details of the diaphragm may be seen in the accompanying views. It is quite similar to the one used at Heidenheim by the J. M. Voith Co. It is 11.33 feet wide, 5.91 feet deep, and weighs 372 pounds. The weight of the car is 330 pounds and the force required to move the car and diaphragm, the latter being unsubmerged, was 1.76 pounds. The clearance between the canal and diaphragm is about 0.4 of an inch. The car runs on ball bearings, two wheels being grooved, the other two flat. A recording device similar to the ones previously described was used to get the velocity of the diaphragm.

In taking a gaging the diaphragm reaches a vertical position in about 7 seconds, and the distance travelled in that time is about 12 feet for the maximum quantity gaged, which was 46 cubic feet per second. The backwater occasioned by the diaphragm subsides rapidly and was so small that it could scarcely be measured. Pulsations in the water are more important and their effect could be seen plainly on the recording chart. The time as recorded by the chronograph was usually checked by personal observation with a stop-watch, both instruments having been calibrated by comparing them with an accurate chronometer.

In order to get a continuous record of the discharge at this station, a recording float-gage, Fig. 26, was installed in the gaging canal. As each gage height corresponds to a certain discharge when the gate at the lower end of the canal is closed, a gage height-discharge curve (see Plate III) was first constructed by means of the diaphragm gagings, and the results applied to the recording float-gage. As may be seen from Fig. 27, the discharge is thus registered directly on the chart with considerable accuracy and the mean discharge over a certain period can be easily determined therefrom. The recording apparatus is housed in a little wooden kiosk, which can be heated in winter so as not to freeze the water in the float pipe or the ink in the recording pen.

Object.—The object of the experiments was to make comparative measurements with current meter, weir, diaphragm, and chemical means.

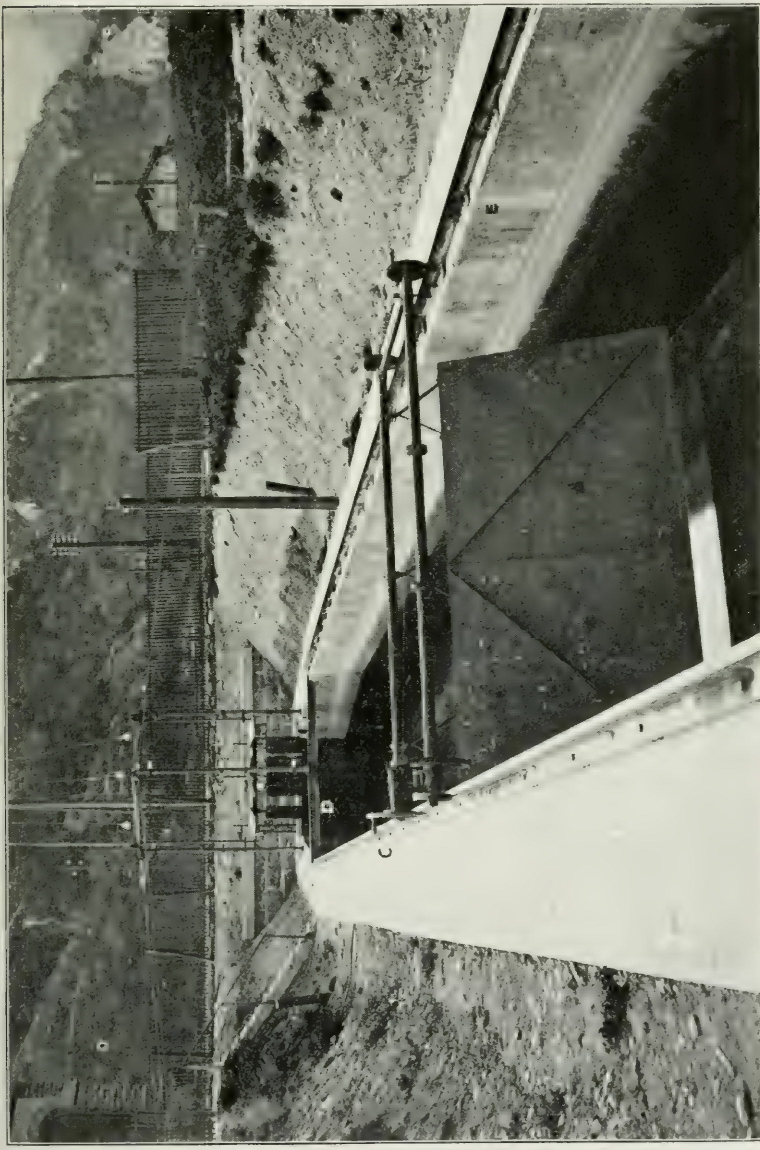

FIG. 23.—UP STREAM VIEW OF RATING CANAL AND DIAPHRAGM AT ACKERSAND.

Current Meter Gagings

The current meter gagings were taken 16.4 feet upstream from the weir (see Fig. 28). A platform was built across the channel so that the meter could always be held in the exact plane of the gaging section. Two benchmarks were established at the section, from which the height of water surface was measured. The following elevations were determined by precise leveling:

B. M. 867	2294.235	feet
B. M. 868	2294.226	"
Mean elevation of bottom of canal	2288.062	"
Mean elevation of weir crest	2291.943	"
Mean elevation of bottom of canal just back of the weir	2288.059	"

The dimensions of the canal and weir crest were:

Mean width of the canal at gaging station	11.427	feet
Mean width of the canal just back of the weir	11.421	"
Mean length of weir crest	11.191	"

Two current meters of the screw type were used in the experiments. Both instruments were very sensitive and began to register at a velocity of about 0.23 of a foot per second, which was necessary in these experiments as the velocities ranged from 0.26 to 1.70 feet per second. The meters were rated before and after the experiments but no difference in the constants was observed.

A few preliminary gagings showed that the velocity was quite uniform over the entire cross-sectional area, and that the vertical velocity curves were symmetrical. In order to get an entire gaging while the flow was constant, it was necessary to restrict the time of the gaging, so that the cross-section was divided into 8 vertical sections and velocity determinations made at either 6 or 7 points in a section. The elevation of the water surface was read at intervals of 5 minutes during a gaging.

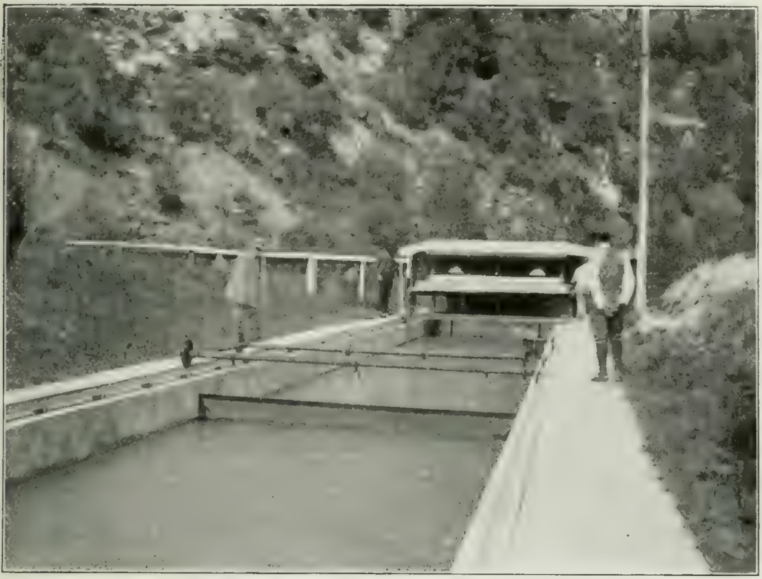

FIG. 24.—VIEW OF DIAPHRAGM DURING A GAGING AT ACKERSAND.

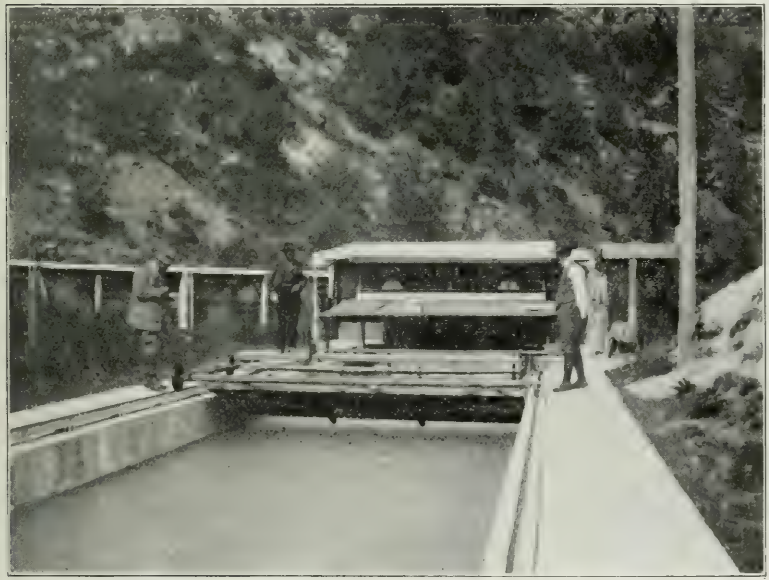

FIG. 25.—VIEW OF DIAPHRAGM AFTER A GAGING AT ACKERSAND.

The results of the current meter gagings have been summarized in the table on Plate V. and in the following one:

CURRENT METER GAGINGS.

Number of gaging.	Number of gaging points. A	Total number of observations.	Mean velocity, in feet per second.	Cross-sectional area, in square feet. F	Area per gaging point, in square feet $\frac{F}{A}$
I	56	157	0.758	57.4195	1.02
II	48	66	0.663	55.8824	1.16
III	48	60	0.787	57.7198	1.40
IV	48	67	0.636	55.3915	1.15
V	48	79	0.626	55.8974	1.16

WEIR GAGING

As previously mentioned a vertical sliding gate at the lower end of the canal was used as a weir. (See Figs. 1, 22, and 28) The crest is sharp and air has access underneath the nappe, but the length is not equal to the width of the canal, as the gate guides on each side cause a slight end contraction of the stream. Moreover, two vertical iron bars for raising the gate cause a further contraction of the stream. The width of the guides is 1.46 inches and that of the gate stems is 2.76 inches, the latter being 1.51 feet from the sides of the canal. The weir was, therefore, not of standard form and the effect of these obstructions on the discharge is rather difficult to estimate correctly. For the conditions existing it was thought best to use the coefficients of discharge as determined by Professor Frese in his exhaustive experiments[4] on weirs with end contractions, although conditions were not exactly similar.

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[4] *Zeitschrift des Vereines deutscher Ingenieure*, Vol. 34, 1890, p. 1285.

FIG. 26.—VIEW OF RECORDING APPARATUS USED AT ACKERSAND.

Gaging by Chemical Means

The gaging of streams by chemical means[5] was proposed a number of years ago, but few experiments were made to determine the accuracy of this method, until those undertaken in connection with the diaphragm experiments by the Swiss Bureau of Hydrography. Full details of these experiments have been published in a bulletin[6] of the Bureau, but an abstract of the method and results is given at this place, as they are of interest for the purpose of comparison with the other methods of gaging.

Method.—If a concentrated solution of some chemical, for which a sensitive reagent is known, be introduced at a uniform rate into a flowing stream and a sample of the mixture be taken at a certain distance downstream, an estimate of the discharge can be made by determining the degree of dilution. Thus, if q_1 represents the quantity of the concentrated solution, and Q_2 the discharge of the stream, then $\frac{Q_2}{q_1} = \frac{k_1}{k_2}$, if k_1 and k_2 represent the degree of concentration. For example, if the concentrated solution contains 300 grams of salt per liter of water ($k_1 = 300$), and one tenth of a liter ($q_1 = 0.1$) per second be discharged into the stream, then $k_1 \times q_1 = 30$ grams of salt will be used each second. Then, if the mixture contains only 0.03 grams of salt per liter of water ($k_2 = 0.03$) the discharge of the stream would be

$$Q_2 = \frac{k_1 \times q_1}{k_2} = \frac{300 \times 0.1}{0.03} = 1000 \text{ liters per second.}$$

To be exact the result should be $Q_2 = 1000 - 0.1 = 999.9$ liters per second, because 0.1 of a liter of the salt solution is discharged into the stream each second.

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[5] *Trans. of Inst. Naval Architects*, Vol. 37, 1896, p. 226.
Proc. of Inst. Civil Engineers, Vol. 160, 1904–05, p. 349.

[6] *Jaugeages par Titrations*, by Dr. Léon W. Collet, Dr. R. Mellet and O. Lütschg. Bern, Switzerland: The Secretary, der Schweizerischen Landeshydrographie. 1.00 Franc.

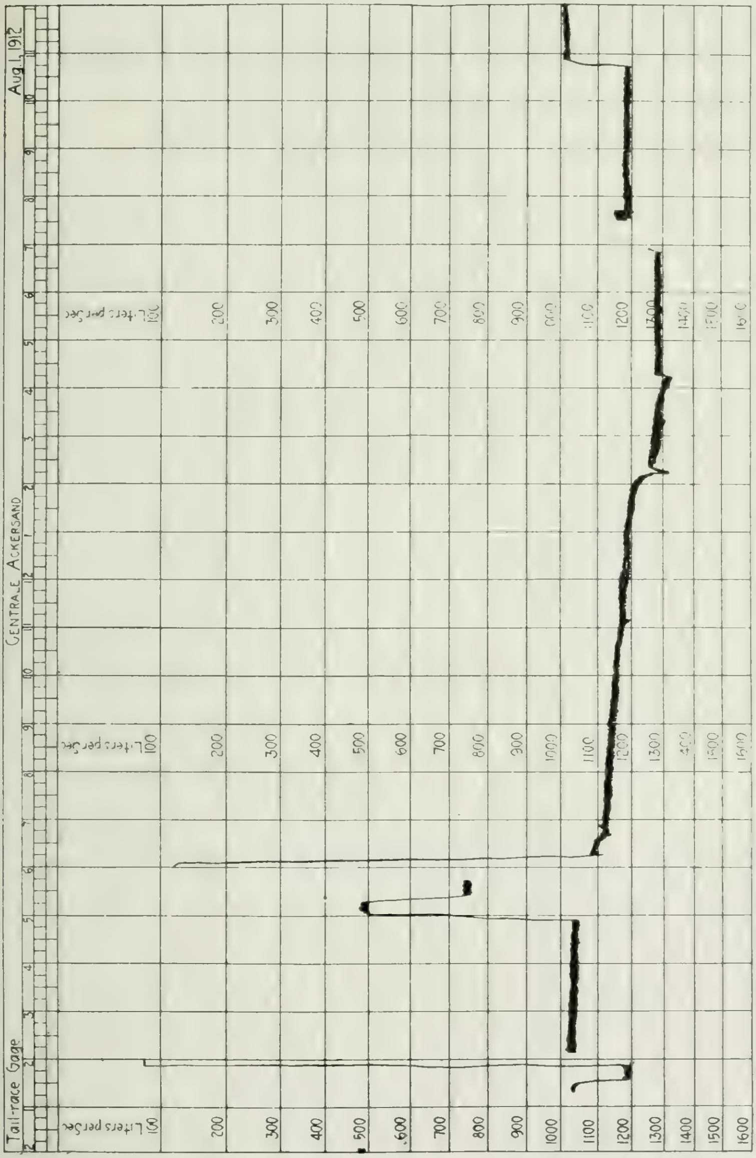

FIG. 27.—REPRODUCTION OF RECORDING GAGE CHART FOR MEASURING DISCHARGE AT ACKERSAND.

For accurate results with this method of gaging the following conditions must be fulfilled:

(1) uniform discharge of the concentrated solution into the stream;

(2) complete mixture of the solution with the stream;

(3) accurate analyses of the concentrated and diluted solutions.

Apparatus.—Fig. 29 shows the apparatus used for discharging the concentrated salt solution into the stream at a uniform rate. The barrel A holds about 200 liters of the solution, the discharge from which is regulated by the cock B and introduced into the stream through the pipe D. In order to keep a constant flow, the cock B is so adjusted that the solution in the cylinder C is always running over the top. The excess discharge is first caught in the container E and then flows into the vessel G, from which it is poured back into the barrel from time to time. The discharge from the pipe D, due to the head H, is, therefore, always constant and the rate of discharge can be determined by a volumetric calibration.

The solution used was made with ordinary table salt. The precaution should be taken to filter it through a piece of cheesecloth, and to see that the vessels in which it is used are clean, so as not to clog the pipe.

A complete mixture of the solution with the stream will depend of course entirely on the existing conditions, more especially on the degree of turbulence of the stream. In power plants it would seem that the mixture would be more complete after it had passed through an impulse wheel than if it had gone through a reaction turbine. The Pelton wheels at Ackersand probably produced very favorable conditions for this method of gaging. In one of the gagings the solution was introduced at the turbine, K, Fig. 28, while in the other it was put in at L, the junction of the tail-race with the main collecting channel. The slope of the tail-race is very steep and the water is in a violent state of commotion as it leaves the power-house, considerable wave action still existing at a distance of 295 feet downstream. Samples of the diluted mixture were taken at M at various points of the cross-section, so that the mixture was fairly

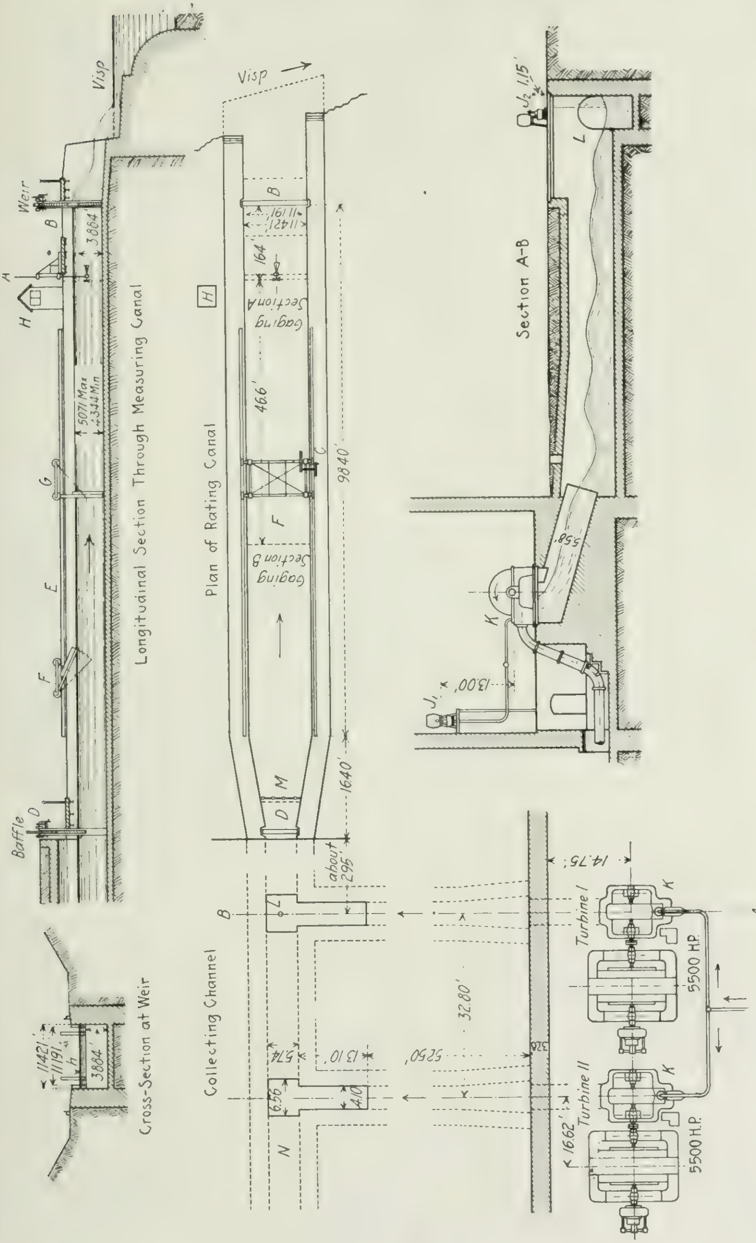

FIG. 28.—PLAN AND SECTIONS OF TAIL-RACE, COLLECTING CHANNEL AND RATING CANAL AT ACKERSAND.

complete for both gagings. Such favorable conditions for mixing seldom exist in streams or in canals where the cross-sectional area is large and the flow fairly uniform. The method would be difficult to apply to such cases; it may, however, be applied to small mountain streams where the velocity is great and the flow more or less turbulent, especially since meter gagings involve considerable difficulties on account of the shallow section and turbulent flow.

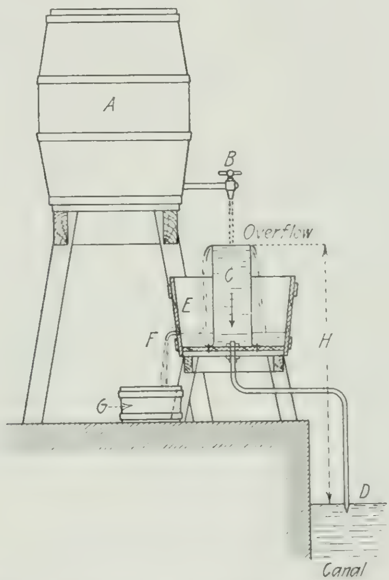

FIG. 29.—CONSTANT DISCHARGE APPARATUS FOR GAGING BY CHEMICAL MEANS AT ACKERSAND.

Chemical Analysis.—The method of making the chemical analysis to determine the amount of salt in the mixture is based on the volumetric titration method and is explained by Dr. Mellet as follows: "The best method of analyzing neutral alkalichloride solutions is that of Mohr, which is based on the use of a titrated silvernitrate solution, using neutral potassium chromate as indicator. The chromate salt produces a lemon colored solution, and the addition of the silver solution occasions the

formation of a white silver chloride precipitate. After all the chloride has been precipitated any excess silver nitrate will react with the indicator and produce a reddish brown silver chromate, which can be detected with the first drop of excessive silver solution by the change in color from lemon to orange." This analysis is made with samples of (1) the concentrated initial solution introduced into the stream; (2) the stream to be gaged; (3) the mixture or final solution. To make the analysis two reagent solutions of the following strengths are needed: about 1.5 to 2 grams of silver nitrate to 1 liter of distilled water, and about 5 grams of potassium chromate to 100 cubic centimeters of distilled water.

Results.—The following table shows the method of procedure and the results obtained in the two gagings made at Ackersand on September 13, 1912.

RESULTS OF GAGINGS BY CHEMICAL MEANS.

1. General data:				
No. of gaging			III A	III B
Time			11-11:20	4-4:07
Duration		min.	20	7
Place (see Fig. 28)			K—M	L—M
No. of turbines being operated			2	2
Mean gage height		ft.	2293.1234	2293.1070
Mean depth of water		ft.	5.071	5.054
2. Concentrated initial salt solution:				
Discharge per second	q_1	liters	0.114525	0.121183
Duration		min.	20	7
Total quantity used		liters	138	51
3. Analysis of stream sample:				
Amount of silver nitrate used to titrate 1 liter	N_0	c.c.	1.6	1.6
4. Analysis of initial salt solution:				
5 cubic centimeters diluted to		c.c.	500	500
Quantity of diluted solution analyzed		c.c.	10	10
Quantity of silver nitrate used in analysis	n_1	c.c.	42.25	43.30
Quantity of silver nitrate used per liter of salt solution	N_1	c.c.	422500	433000
5. Analysis of final mixture:				
6 samples were taken after a period of		min.	7 to 20	5 to 8
Average amount of silver nitrate used to titrate 1 liter	N_2	c.c.	37.2675	41.8000
6. Results:				
Discharge $Q_2 = q_1 \dfrac{N_1}{N_2 - N_0} - q_1$	Q_2	liters	1356	1305
Degree of dilution $\dfrac{Q_2}{q_1} = \dfrac{k_1}{k_2}$			11800	10800

Results of Gaging I

In Gaging I a series of 12 runs was made with the diaphragm from 9:14 to 9:45 A. M., before the current meter gaging was taken. The following data were observed:

Diaphragm Gaging I—Before Current Meter Gaging.
Time 9:14 to 9:45 A. M., May 3, 1910.

No. of run.	Mean Velocity V' in the Canal Measured Over a Distance of 46.035 Feet According to:	
	Observations with stop-watch.	Readings from recording chart.
1	0.7873 ft. per sec.	0.7710 ft. per. sec.
2	0.7742	0.7742
3	0.7808	0.7742
4	0.7742	0.7742
5	0.7808	0.7742
6	0.7808	0.7742
7	0.7742	0.7742
8	0.7742	0.7710
9	0.7808	0.7775
10	0.7742	0.7710
11	0.7676	0.7742
12	0.7808	0.7742
Average value of V'I.	0.7775	0.7748

After the current meter gaging was finished, a second series of diaphragm runs was made from 11:39 A. M. to 12:04 P. M., the following results being obtained:

DIAPHRAGM GAGING I—AFTER CURRENT METER GAGING.
TIME 11:39 A. M. TO 12.04 P. M., May 3, 1910.

No. of run.	Mean Velocity V″ in the Canal Measured over a Distance of 46.035 Feet According to:	
	Observations with stop-watch.	Readings from recording chart.
1	0.7742 ft. per sec.	0 7742 ft. per sec.
2	0.7676	0.7742
3	0.7808	0.7808
4	0.7808	0.7808
5	0.7742	0.7742
6	0.7808	0.7710
7	0.7676	0.7742
8	0.7742	0.7742
9	0.7742	0.7775
10	0.7742	0.7676
11	0.7742	0.7775
12	0.7808	0.7808
Average value of V″ I.	0.7753	0.7768

A constant discharge being maintained throughout gaging I, the mean of the two series of runs may be taken for later comparison,

$$\frac{V'I+V''I}{2} = \frac{0.7775 + 0.7753}{2} = 0.7764 \text{ ft. per. sec.},$$

the values of the velocity as observed with the stop-watch being used as these were considered more accurate.

In gaging I, due to a defect in the apparatus, the diaphragm did not come to a vertical position but stood at an angle of 21 degrees with the vertical. Consequently the diaphragm was only effective in measuring the mean velocity in a part of the cross-sectional area. If the actual cross-sectional area of 57.4195

square feet be multiplied by the above mean value of the velocity, the computed discharge

$$57.4195 \times 0.7764 = 44.575 \text{ cu. ft. per sec.},$$

will evidently be too large because a considerable strip (6.8427 square feet or 12 per cent of the total area) of the water cross-section is not taken into consideration; a strip in which just the smallest velocities occur. If, on the other hand, the area of the vertical projection of the diaphragm, 50.5768 square feet, be multiplied by the same mean velocity, the result would evidently be too small because the water flowing beneath the diaphragm has not been included. In order, therefore, to compare the diaphragm gaging with the current meter gaging, an adjustment of the data is necessary.

The current meter gaging was made between 10:01 and 11:29 A. M. The velocity was observed at 56 points in the cross-section and the meter held at each point about 90 seconds, three observations being taken during that time. As a result of this gaging a discharge of 43.505 cubic feet per second was obtained, which corresponds to a mean velocity of 0.7578 feet per second. An examination of the vertical velocity and the mean velocity curves (not shown herewith) showed that the velocities were quite uniform over the section, with the exception of a strip 0.82 of a foot in depth at the bottom and the two sections next to the side walls. The velocities within these sections only vary between 0.738 and 0.903 of a foot per second. If the lower part of the vertical velocity curves, corresponding to the area not included in the diaphragm gaging, be cut off and the remainders used to compute the mean velocity, the result is 0.784 feet per second, and the corresponding discharge 39.69 cubic feet per second. This mean velocity is then 0.784 — 0.766 = .008 of a foot higher than that obtained with the diaphragm. In this gaging it was not possible to put the diaphragm in a vertical position.

Results of Gaging II

Under the general heading of gaging II, several experiments were made with the object of checking the diaphragm gaging with the current meter gaging, under the following conditions: (1) diaphragm in a vertical position; (2) diaphragm at various angles of inclination with the vertical; (3) diaphragm in a vertical position and perturbed condition of flow.

Diaphragm in a Vertical Position.—Two series of ten runs each, one before and the other after the current meter gaging, were taken with the diaphragm in a vertical position. The observed data are given in the following tables:

Diaphragm Gaging II Before Current Meter Gaging.
Time 2:32 to 2:59 P. M., August 1, 1912.

No. of run.	Mean Velocity V′ in the Canal Measured Over a Distance of 32.8 Feet According to:	
	Observations with stop-watch.	Readings from recording chart.
1	0.6561 ft. per sec.	0.6594 ft. per sec.
2	0.6594	0.6594
3	0.6626	0.6660
4	0.6626	0.6660
5	0.6626	0.6660
6	0.6660	0.6660
7	0.6496	0.6594
8	0.6561	0.6594
9	0.6692	0.6660
10	0.6692	0.6758
Average value of V′II.	0.6614	0.6643

DIAPHRAGM GAGING II—AFTER CURRENT METER GAGING.
TIME 4:25 TO 4:44 P. M., AUGUST 1, 1912.

No. of Run.	Mean Velocity V'' in the Canal Measured over a Distance of 32.8 Feet According to:	
	Observations with stop-watch.	Readings from recording chart.
11	0.6626 ft. per sec.	0.6594 ft. per sec.
12	0.6692	0.6758
13	0.6660	0.6692
14	0.6626	0.6660
15	0.6626	0.6692
16	0.6692	0.6692
17	0.6626	0.6725
18	0.6660	0.6692
19	0.6660	0.6725
20	0.6692	0.6725
Average value of V'' II.	0.6656	0.6696

The discharge remaining constant throughout both series, the mean of the average values may again be taken for later comparison,

$$\frac{V'II+V''II}{2} = \frac{0.6614+0.6656}{2} = 0.6635 \text{ ft. per sec.}$$

Current meter gaging II was performed between 3:03 and 4:20 P. M. Readings were taken at 48 points in the section, the meter being held at least 40 seconds at each point. As a constant load on the turbines could only be maintained until towards evening, the current meter gaging had to be restricted to as short a time as possible. The meter was, therefore, read only once or twice at each point.

A discharge of 37.113 cubic feet per second was obtained with the meter gaging. With a cross-sectional area of 55.8824 square feet, the corresponding mean velocity is $V = 0.6626$ feet per second. The difference between the diaphragm gaging and the

meter gaging is 37.148 – 37.113 = 0.035 cubic feet per second, or one tenth of 1 per cent. This is inded a very close check.

Diaphragm at Various Angles of Inclination with the Vertical.—The results of gaging I, in which the diaphragm was inclined at an angle of 21° with the vertical, showed that the diaphragm gaging checked the meter gaging within about 1 per cent, when those parts of the vertical velocity curves not touched by the diaphragm were neglected in computing the mean velocity. This striking fact induced the experimenter to investigate still further the effect of the angle of inclination. Accordingly three more runs were made with the diaphragm at different angles of inclination. The results of these runs, adjusted similarly to the method used for gaging I and compared with meter gaging II, are given in the following table:

	Run 21—5 P. M. Aug. 1, 1912. Diaphragm inclined at an angle of 10°52′ with vertical.		Run 22—5:03 P. M. Aug. 1, 1912. Diaphragm inclined at an angle of 17°27′ with vertical.		Run 23—5:05 P. M. Aug. 1, 1912. Diaphragm inclined at an angle of 32°36′ with vertical.	
	Meter.	Diaphragm.	Meter.	Diaphragm.	Meter.	Diaphragm.
V (ft).......	0.6692	0.6496	0.6758	0.6430	0.6791	0.6200
F (sq. ft.)..	53.66	53.66	51.04	51.04	41.61	41.61
Q (cu. ft. per sec.)......	35.947	34.852	34.464	32.803	28.250	25.812

The above data may be compared directly as the flow was constant throughout the experiments. These show that, with an increase in the angle of inclination, the velocity as measured with the diaphragm decreases, and that the disparity between meter and diaphragm gagings increases.

Diaphragm in a Vertical Position and Perturbed Condition of Flow.—After obtaining such a close check between meter and diaphragm gagings, the question arose whether an agreement equally good would have been obtained if the vertical velocity and mean velocity curves had been greatly distorted. In order to produce such a condition the weir gate at the lower end of the canal was raised, first 0.1 of a foot (runs 24 and 25) and then

0.2 of a foot (runs 26 and 27). Trial vertical velocity curves were then obtained at station 2.00 for both gate openings, to determine the distortion. These curves are shown in Fig. 30. As may be seen the variation in velocity was not obtained in the degree desired. Nevertheless it is interesting to compare the re-

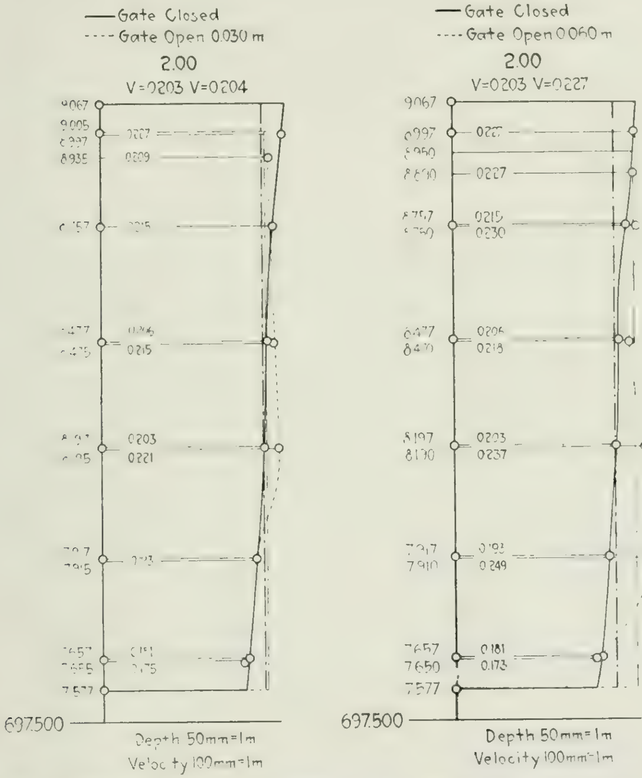

FIG. 30.—VERTICAL VELOCITY CURVES, GAGING II, SWISS BUREAU OF HYDROGRAPHY EXPERIMENTS.

sults. Inasmuch as the flow was constant during all of these experiments, the results may again be compared with meter gaging II without further adjustment. The following table shows that meter and diaphragm gagings under these conditions check almost exactly.

	Weir Gate Raised 0.10 feet				Weir Gate Raised 0.20 feet			
	Run 24 5:14 P. M., Aug. 1, 1912		Run 25 5:16 P. M., Aug. 1, 1912.		Run 26 5:23 P. M., Aug. 1, 1912.		Run 27 5:25 P. M., Aug. 1, 1912.	
	Meter	Diaphragm	Meter	Diaphragm	Meter	Diaphragm	Meter	Diaphragm
V (ft.).........	0.6856	0.6987	0.6889	0.6922	0.7151	0.7184	0.7217	0.7250
F (sq. ft.).....	54.12	54.12	53.97	53.97	51.87	51.87	51.49	51.49
Q (cu. ft. per sec.)............	37.11	37.82	37.11	37.36	37.11	37.25	37.11	37.32

Effect of Wave Action on Diaphragm Gagings.—In order to determine the effect of wave action on diaphragm gagings, 6 runs were made; three with the baffle D, Fig. 23, in place, and three with the baffle removed. The baffle can be raised or lowered and when at its lowest travel it is submerged to a depth of 2.3 feet. With the baffle in this position the waves are almost entirely eliminated; the amplitude of the waves at the gaging section during the foregoing experiments was only 0.007 of a foot. If, however, the baffle be removed, the waves reach a height of 0.7 of a foot. The following table gives the results of these runs:

Diaphragm Gagings, 10:30 to 10:48 A. M., Aug. 2, 1912.

	With Baffle Submerged 2.3 Feet.			Without Baffle.		
	Run 28.	Run 29.	Run 30.	Run 31.	Run 32.	Run 33.
V (ft.)........	0.7906	0.8037	0.7906	0.8004	0.8037	0.8037
F (sq. ft.)....	57.38	57.38	57.38	57.38	57.38	57.38
Q (cu. ft. per sec.).........	45.375	46.115	45.375	45.940	46.115	46.115

The waves apparently have an accelerating effect on the movement of the diaphragm, although this cannot be definitely determined from these limited data.

Results of Gaging III

The object of the experiments, under the general heading of gaging III, was primarily to test the accuracy of the method of gaging by chemical means, by comparison with meter and diaphragm gagings. As in the previous gagings, a series of ten runs each was made with the diaphragm before and after the meter gaging. The accompanying tables show the results obtained.

Gaging III, with current meter, was taken between 11:18 A. M. and 12:22 P. M., and the discharge obtained was 45.518 cubic feet per second. The variations in gage height, vertical velocity curves, and mean velocity curve are shown on Plate I. With a cross-sectional area of 57.7198 square feet, the corresponding mean velocity is 0.7872 feet per second. During the first series of diaphragm runs the gage height was 2293.1201 feet, while during the meter gaging and second series of diaphragm runs it was 2293.1005 feet. The result of the latter and the meter gaging can, therefore, be compared directly.

Diaphragm Gaging III	Meter Gaging III
V = 0.7935 ft. per sec.	V = 0.7872 ft. per sec.
F = 57.7198 sq. ft.	F = 57.7198 sq. ft.
Q = 45.835 cu. ft. per sec.	Q = 45.518 cu. ft. per sec.

The two methods of gaging check within 0.7 of 1 per cent.

Gagings IIIA and IIIB by Chemical Means.—Gagings IIIA and IIIB were made with the chemical method, using a salt solution as the gaging fluid. IIIA was made at 11:00 A. M., with a gage height of 2293.1234 feet, the solution being introduced into the jet operating the turbine. IIIB was made at 4 P. M., with a gage height of 2293.1070 feet, the solution being put into the stream at the junction of the tail-race of turbine No. 1 with the main collecting channel. (See Fig. 28). Diaphragm gagings were made during both of these gagings, but simultaneous meter gagings could not be made, as they would have

DIAPHRAGM GAGING III—BEFORE CURRENT METER GAGING.
TIME 10:56 TO 11:15 A. M., SEPTEMBER 13, 1912.

No. of run.	Mean Velocity V′ in the Canal Measured over a Distance of 32.8 Feet According to:	
	Observations with stop-watch.	Readings from recording chart.
1	0.7972 ft. per sec.	0.7972 ft. per sec..
2	0.7972	0.8004
3	0.7972	0.7972
4	0.7873	0.7906
5	9.7939	0.7775
6	0.7873	0.7873
7	0.7906	0.7939
8	0.7906	0.7939
9	0.7972	0.7972
10	0.7906	0.7939
Average value of V′ III.	0.7929	0.7929

DIAPHRAGM GAGING III—AFTER CURRENT METER GAGING.
TIME 12:50 to 1:12 P. M., SEPTEMBER 13, 1912.

No. of run.	Mean Velocity V″ in the Canal Measured over a Distance of 32.8 Feet According to:	
	Observations with stop-watch.	Readings from recording chart.
11	0.7972 ft. per sec.	0.8004 ft. per sec.
12	0.7972	0.7972
13	0.7906	0.7972
14	0.8037	0.8070
15	0.7939	0.7972
16	0.7873	0.7972
17	0.7873	0.7972
18	0.7873	0.7972
19	0.7972	0.7972
20	0.7939	0.7972
Average value of V″ III.	0.7935	0.7985

taken too long a time. In order to get some comparison with a meter gaging, a discharge curve, Plate III, was plotted from the results of previous meter gagings, from which the discharges were picked off for the corresponding gage heights. The comparative results are shown in the following tables:

GAGING III A—GAGE HEIGHT 2293.1234 FEET.

	Gaging by chemical means	Diaphragm gaging, average of 8 runs	Meter gaging, from curve
V (ft.)	....	0.7939	....
F (sq. ft.)	....	57.97	....
Q (cu. ft. per sec.)	47.885	46.012	46.507

GAGING III B—GAGE HEIGHT 2293.1070 FEET.

	Gaging by chemical means	Diaphram gaging, average of 10 runs	Meter gaging, from curve
V (ft.)	....	0.7939	....
F (sq. ft.)	....	57.79	....
Q (cu. ft. per sec.)	46.082	45.870	45.590

The comparatively large discrepancy between gaging IIIA by chemical means and the other methods may be explained partly by the fact, that in gaging IIIA, in which the salt solution was introduced at the turbines, the final mixture was somewhat diluted by the stagnant water (at N, Fig. 28) existing in the upper part of the collecting channel. The length of this channel is about 100 feet, and will collect the discharge from three turbines which were not yet installed. The effect of this dilution would make the discharge too great. In gaging IIIB, in which the salt solution was introduced at L, Fig. 28, the stagnant water in the channel did not come in contact with the final mixture, being cut off by the flow from turbine II.

Results of Gagings IV and V

In the experiments performed thus far the conditions of flow were such that the velocities throughout the cross-section were quite uniform. The attempt was made in gaging II to produce a greater variation by raising the gate at the lower end of the canal, but this did not, however, produce the desired result. In gagings IV and V a perturbed condition of flow was obtained by placing an obstruction of boards, 7 feet wide, at the right wall of the canal, 74 feet up-stream from the gaging section, so as to deflect the stream toward the left wall. Two complete meter gagings were then made at two different sections of the canal, in order to determine the distribution of velocities. Gaging IV was made at the regular gaging section (section A), Fig. 28, 16.4 feet upstream from the weir, between 11:37 A. M. and 2:08 P. M. Gaging V was made at a section (section B), 46.6 feet upstream from section A., between 2:15 and 3:32 P. M. The vertical velocity and mean velocity curves for both of these gagings have been plotted, and are shown on Plate II. The following table also shows the maximum and minimum velocities, as measured with the meter, for all of the gagings:

No. of gaging.	Minimum velocity, in feet per second.	Maximum velocity, in feet per second.	Difference, in feet per second.
I	0.387	0.902	0.515
II	0.499	0.764	0.265
III	0.541	0.941	0.400
IV (section A)	0.249	0.905	0.656
V (section B)	−0.472	1.699	2.171

From the figures above and the curves on Plate II, it may be seen that the flow was perturbed, and that the velocities varied considerably. This condition of flow now being established, two series of runs were made with the diaphragm. The first series

with the obstruction in the canal consisted of 5 runs, and was made between 3:37 and 3:47 P. M. In the second series the obstruction was removed and 5 runs again taken between 3:54 and 4:08 P. M. The difference in gage heights at the gaging section for the two series was only 0.0066 of a foot, This small difference may be neglected in comparing the results, considering that the obstruction, no doubt, affected the gage height. The results of the two series of diaphragm gagings are:

Mean velocity with the obstruction in place, average of 5 runs, 0.6364 feet per second.

Mean velocity with the obstruction removed, average of 5 runs, 0.6397 feet per second.

The conclusion may, therefore, be drawn that the diaphragm will give equally good results with perturbed flow as with uniform flow.

The difference in the two meter gagings was 0.5 of 1 per cent. This close check under these conditions was due to the extreme care used in performing the gagings, and to the precise rating of the meter. The negative velocities at section B could be plainly seen and measured, as the water was very clear. More information in regard to the meter gagings may be found on Plates II and V, which need no further explanation.

Discussion of the Results

The results of the different gagings have been plotted and are shown on Plate III. The different methods of gaging have been distinguished by using different characters for the plotted points. The solid curve represents the average of the plotted meter gagings, while the dashed curve is the computed weir-discharge curve, using Frese's coefficients. The latter curve at first glance appears abnormal because it crosses the meter discharge curve. This anomaly may, perhaps, be partly due to the contractions caused by the two stems for raising the weir, the effect of which seems to vary with the head.

On Plate VI a summary of the experimental and computed discharges, with their absolute and relative differences, has been tabulated. From this table and the curves on Plate III, it can

be seen without further discussion, that the diaphragm gagings check the meter gagings very closely. The differences range between such small limits, that for most practical purposes they are of no significance. The greatest difference obtained with a direct comparison of meter and diaphragm gagings was 0.7 of 1 per cent. Theoretically the results of the diaphragm gagings should be too small rather than too large, as the velocity of the diaphragm is retarded by the frictional resistance of the car. It seems, however, that the effect of this resistance is offset by the smaller velocities, existing in the clearance between the diaphragm and the periphery of the canal, which are not measured by the diaphragm.

The smallest quantity gaged in the rating canal at Ackersand was 1.24 cubic feet per second, at which rate of discharge the diaphragm still operated faultlessly. It is to be noted particularly, that the length of the gaging distance should be long enough to overcome the effect of pulsations in the stream, for, as may be seen on Plate IV, the velocity of the diaphragm was not uniform over the entire gaging distance.

CONCLUSION

The results of the experiments described may be summarized in the following conclusions:

1. The diaphragm gagings agree with the meter gagings within 1 per cent.
2. The rating canal for diaphragm gagings should be of sufficient length to overcome the effect of pulsations in the stream.
3. The diaphragm should be placed in a vertical position in making a gaging.
4. The diaphragm method of gaging will give equally good results with perturbed flow as with parallel flow.
5. Under certain conditions gaging by chemical means affords an accurate method of measuring discharge, the results of the experiments checking the meter gagings within 3 per cent.

There is great need at the present time for a device for measuring moderately large quantities of water, which shall be simple, accurate, rapid and, at the same time, inexpensive. For the successful operation of an irrigation system, the measurement of the water is essential, and each season brings new demands for an accurate system of measurement It is not likely that any device can supersede the weir for the measurement of the flow in small laterals, but for the larger canals the diaphragm method possesses advantages, which in some cases would make its use preferable to either the weir or current meter. In many channels there is not sufficient fall to permit the use of the weir, and in others the silt-bearing streams clog up the channel of approach, so as to affect seriously its accuracy.

There will always be a large number of cases in practice, where no other method of measuring discharge will be available except by current meter, especially when dealing with short canals and large quantities of water. Where conditions are suitable, the diaphragm has, however, this distinct advantage over the current meter, namely, the rapidity with which a gaging can be made. Rating flumes can be easily calibrated, and some form of recording device, similar to the one at the Lonza station, installed to give a record of the total flow.

In industrial lines also, especially in the development of water power, where it is of great importance to the turbine manufacturer, as well as the owners of water power, to be able to test quickly and without much expense the efficiency of turbines, the diaphragm affords an excellent means of measuring discharge. In the design of power plants attention should, therefore, be given to the possibility of providing a suitable canal and installing the necessary apparatus.

The application of the diaphragm method is obviously limited, but no one method will apply to all situations. Past experience with the diaphragm in Europe has been very satisfactory, and it will undoubtedly come into use in this country, where suitable canals are available and the cost of installation is not too great. In view of its advantages and accuracy, the diaphragm method, therefore, deserves recognition as a valuable contribution to the science of hydrometry.

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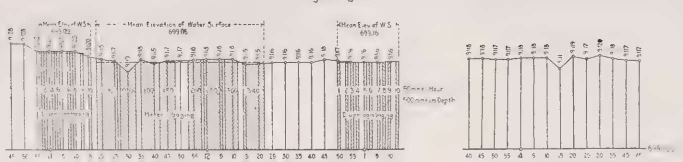
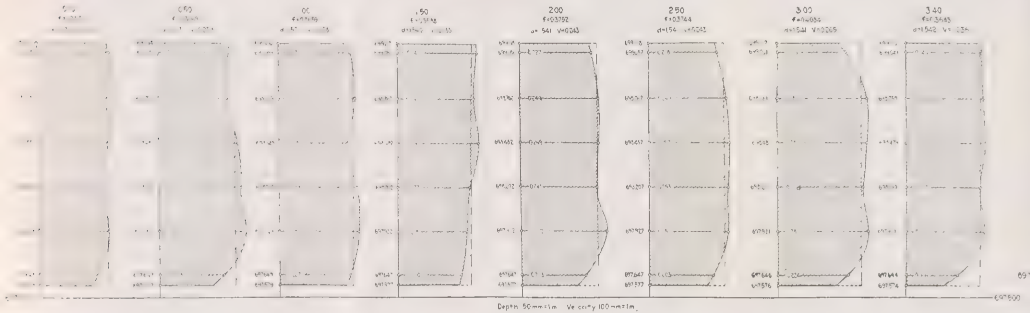
Swiss Bureau of Hydrography Experiments

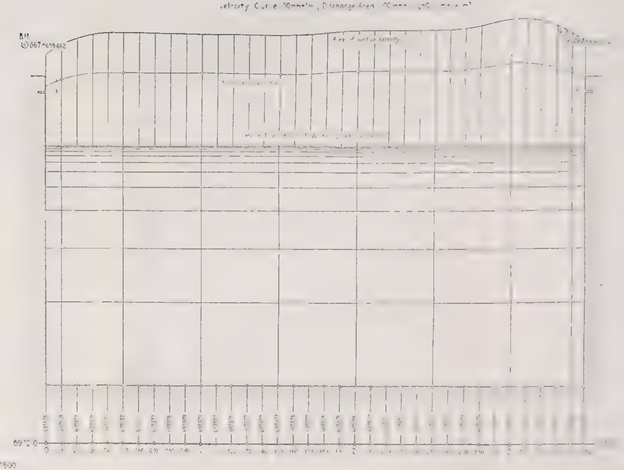
Gaging III

Variation of Gage Height



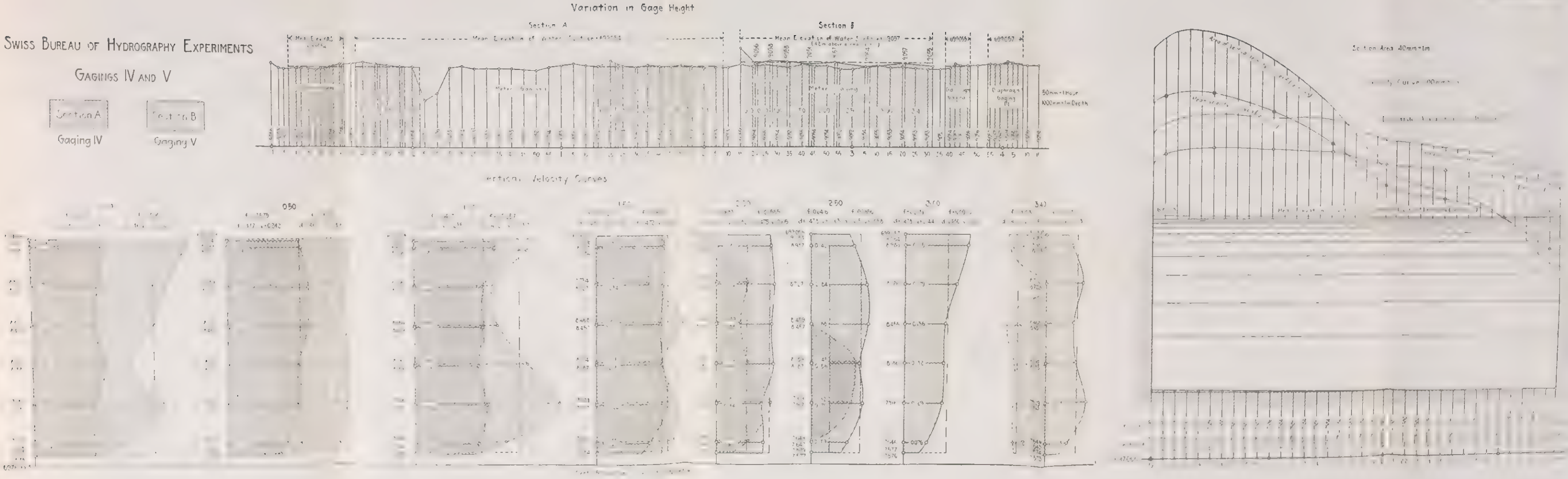
Vertical Velocity Curves





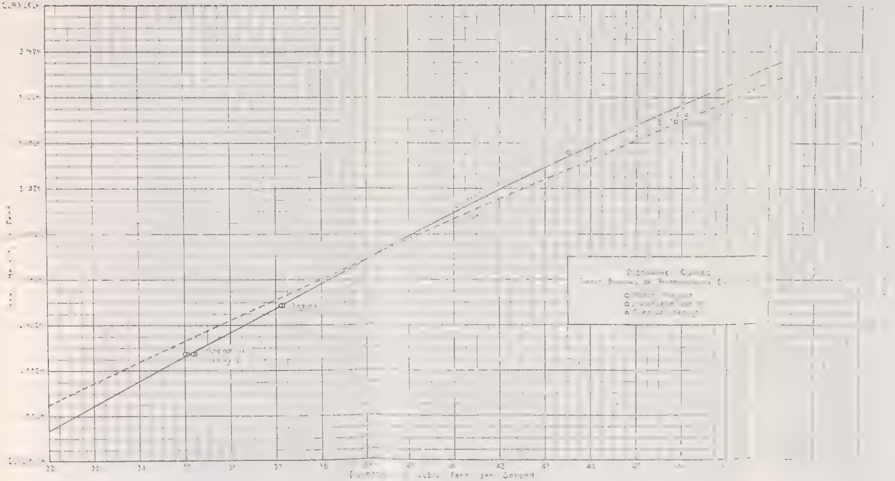
72^2

PLATE II
WEIGNER ON
DIAPHRAGM METHOD
FOR MEASUREMENT OF WATER



12^3

PLATE III
WEIDNER ON
DIAPHRAGM METHOD
FOR MEASUREMENT OF WATER



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RESULTS OF GAGING I WITH THE DIAPHRAGM

SWISS BUREAU OF HYDROGRAPHY EXPERIMENTS

READINGS FROM THE PRECISION CHART

Distances Between Contact Points*		Run 1		Run 2		Run 3		Run 4		Run 5		Run 6		Run 7		Run 8		Run 9		Run 10		Run 11		Run 12		
[illegible]	[illegible]	Reading	Velocity	Reading	Velocity	Reading	Velocity	Reading	Velocity	Reading	Velocity	Reading	Velocity	Reading	Velocity	Reading	Velocity	Reading	Velocity	Reading	Velocity	Reading	Velocity	Reading	Velocity	
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	First series [illegible] Average value of V1 = 0.7810 [illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	Second series 11 [illegible] to 12.04 P. M. Average value of V1 = 0.7834 [illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	First series Ave. value 0.7775 ft. sec.
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	Second series Ave. value 0.7752 ft. sec.
[illegible]	[illegible]	Reading	Velocity	Reading	Velocity	Reading	Velocity	Reading	Velocity	Reading	Velocity	Reading	Velocity	Reading	Velocity	Reading	Velocity	Reading	Velocity	Reading	Velocity	Reading	Velocity	Reading	Velocity	
[illegible]		Run 1		Run 2		Run 3		Run 4		Run 5		Run 6		Run 7		Run 8		Run 9		Run 10		Run 11		Run 12		

Observations with the [illegible] Watch

*Corrected Chronometer Readings

[illegible]

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RESULTS OF CURRENT METER GAGINGS

SWISS BUREAU OF HYDROGRAPHY EXPERIMENTS

		Results					Remarks.
1	No. of gaging	I	II	III	IV	V	
2	Date	May 3 1910 10 01 to 11 29 A M	Aug 1, 1912 [illegible] to 4 20 P M	Sept 13 1912 11 18 A M to 12 22 P M	Dec 11, 1912 11 37 A M to 12 03 P M 1 13 to 2 08 P. M	Dec 11, 1912 2 15 to 3 32 P. M	Gagings I, II, III, and IV were made at section [illegible] A [illegible] ft upstream from the [illegible] while gaging V was made at section [illegible] 46 feet upstream from section A
3	Mean gage height at gaging section in feet	2202 0742	[illegible]	22[illegible]3 1007	2202 8971	2202 9398	
4	Change in gage height during the gaging in feet	0 0164	[illegible]	0 [illegible]	[illegible]	0 [illegible]	
5	Width of gaging section in feet	11 4[illegible]	11 4[illegible]	11 42[illegible]	11 429	11 42[illegible]	
6	Mean depth at gaging section in feet	[illegible] 323	[illegible]	5 0085	4 85[illegible]3	4 8782	
7	Area of gaging section in square feet	57 41[illegible]	[illegible]	57 7198	55 [illegible]	55 8974	
8	Discharge in cubic feet per second	43 505	[illegible] 113	45 [illegible]18	35 172	34 965	
9	Maximum velocity in feet per second	0 9021	[illegible]	0 8922	0 9054	1 6991	
10	Minimum velocity in feet per second	0.3871	0 [illegible]	0 5413	0 2403	−0 4724	
11	Mean velocity in feet per second	0 7578	[illegible]	0 7872	0 6364	0 6266	
12	Ratio of mean to maximum velocity	0 840	[illegible]867	0 882	0 703	0 369	
13	Wetted perimeter in feet	21,487	[illegible]	21,519	21 126	21 158	
14	Hydraulic radius in feet	2 673	[illegible]	2 690	2 624	2 657	
15	Mean temperature of water in degrees Fahrenheit	34 2	[illegible]	40 6	35 6	35 6	
16	Mean temperature of air in degrees Fahrenheit	48 4	[illegible]	45 7	26 6	26 6	

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RESULTS OF THE EXPERIMENTAL AND COMPUTED DISCHARGES, SWISS BUREAU OF HYDROGRAPHY EXPERIMENTS

Mean Elevation of Weir Crest 2291 9126 Feet

Number of test	Date	Measured discharge in cu. ft. per sec.: Meter Q	Diaphragm Q'	Chemical Q''	Mean elevation of water surface above weir crest in feet above sea level	Corresponding head on weir in feet	Computed discharge in cu. ft. per sec. using Frese's coefficients Q*	Differences between the metered discharge Q and the discharges obtained with the diaphragm Q' and chemical means Q'': Absolute in cu. ft. per sec. (Q' − Q)	Relative in % $\left(100 \frac{Q' - Q}{Q'}\right)$	Differences between the computed and measured discharges: Absolute in cu. ft. per sec. (Q* − Q)	Relative in % $\left(100 \frac{Q^* - Q}{Q^*}\right)$
[illegible]	[illegible] 1 [illegible] 1[illegible] to 11 2[illegible] A. M.	4[illegible]			[illegible]	1.1316	44 [illegible]88	Diaphragm Chemical	Diaphragm Chemical	+0 [illegible] Meter [illegible] Diaphragm Chemical	+2 [illegible] Meter Diaphragm Chemical
I[illegible]	[illegible] 1 1912 [illegible] to 4 [illegible]0 P. M.	[illegible]	37.148		[illegible]	[illegible]	[illegible]	+0 [illegible] Diaphragm Chemical	+0 1 Diaphragm Chemical	[illegible] Meter [illegible] Diaphragm Chemical	1 1 Meter 1 2 Diaphragm Chemical
III	Sept 13 1912 11 18 A. M. to 12 22 P. M.	4[illegible] 518	45 8[illegible]		[illegible]	1 17[illegible]	4[illegible] 842	+0.317 Diaphragm Chemical	+0 6 Diaphragm Chemical	[illegible] Meter +[illegible] Diaphragm Chemical	+[illegible] Meter [illegible] Diaphragm Chemical
IIIA	Sept 13 1912 11 [illegible] A. M.	46 507	46 [illegible]13	47 8[illegible]	[illegible]	1.1808	47 283	+0.404 Diaphragm +1.378 Chemical	[illegible]1 1 Diaphragm +2 9 Chemical	+0 77[illegible] Meter −1 27[illegible] Diaphragm −0 [illegible] Chemical	+1 6 Meter +2 7 Diaphragm 1.3 Chemical
IIIB	Sept 13, 1912 4 [illegible] P. M.	45 [illegible]	45 870	46 082	[illegible] 1070	1 1644	46.330	+0 280 Diaphragm +0 4[illegible]2 Chemical	+0 6 Diaphragm +1 1 Chemical	+0 [illegible] Meter 0 46[illegible] Diaphragm +0 248 Chemical	[illegible]1 6 Meter −1 0 Diaphragm +0 5 Chemical
IV	Dec 11 1912 11 57 A. M. to 12 03 P. M. 1 13 to 2 [illegible] P. M.	[illegible] 17[illegible]	35 068		[illegible]92 8971	0 9545	34 429	−0 104 Diaphragm Chemical	−0.3 Diaphragm Chemical	0 743 Meter [illegible] Diaphragm Chemical	[illegible]2 2 Meter [illegible]1 8 Diaphragm Chemical
V	Dec 11, 1912 2 17 to 3 32 P. M.	34 995	35 242		[illegible]92 8971	0 9545	34 429	−0 247 Diaphragm Chemical	+0 7 Diaphragm Chemical	0 566 Meter [illegible]81[illegible] Diaphragm Chemical	1 6 Meter −2 4 Diaphragm Chemical

BULLETIN OF THE UNIVERSITY OF WISCONSIN

NO. 699

ENGINEERING SERIES, VOL. 8, NO. 2, PP. 73-146

THE FLOW OVER WEIRS WITH IMPERFECT CONTRACTIONS

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MADISON, WISCONSIN

1914

[75]

CONTENTS

PREFACE

This bulletin is based upon the results of experiments made in the Hydraulic Laboratory of The University of Wisconsin by students of the College of Engineering, working under the direction of the writer. The experiments were begun in the fall of 1906 by A. S. Diehl and F. C. Ebert, who experimented with a 12-inch weir with channels of 26½-in. and 18-in. depths and various widths. During 1907–8 the experiments were continued by G. S. Cortelyou and E. L. Hain, who used the same apparatus, making the depths of the channels 12 in. and 6 in. Experiments with an 18-in. weir were made by R. F. Storer and L. L. Ladd during the summer of 1908, and with a 9-in. weir by H. Swenholt and C. T. Dahl during 1908–9. The results of these experiments were presented by their authors as theses for the bachelor's degree.

The experiments herein described were conducted on a comparatively small scale, with heads up to only about 0.6 ft., by inexperienced observers. The data, however, seem to be quite consistent and form a valuable contribution to the meager information on this important subject.

THE FLOW OVER WEIRS WITH IMPERFECT CONTRACTIONS

INTRODUCTION

The ideal conditions of flow over a weir occur when the weir is situated in the side of a deep wide reservoir. Under such conditions the contraction is perfect and complete and the velocity of approach is negligible. The most usual location of a measuring weir, however, is across a channel of relatively narrow width, and the question arises as to how to construct the weir to conduce to the most reliable measurements. Such conditions make it doubtful whether the bottom and end contractions of the sheet of water are perfect and whether the effect of the velocity of approach is negligible. As the result of a careful study of all published results available at the time, the late Hamilton Smith, Jr., concluded that if the crest of the weir were three times the maximum head above the bottom of the channel and the ends of the notch twice the maximum head from the sides of the channel the contractions would be practically perfect. These specifications call for dimensions of channel which often cannot be obtained. Under such conditions it has been considered safest to suppress the end contractions entirely by making the length of the weir equal to the width of the channel. It is important to have some means of estimating the error due to reducing the depth of the channel from the values specified by Hamilton Smith, Jr., because it is impracticable in many cases to have so great a depth. There are many instances, also, where weirs are built in such a way as to have imperfect end contractions. The experiments described in this bulletin were made with the purpose of adding information on this subject.

DESCRIPTION OF EXPERIMENTAL APPARATUS

The apparatus used in these experiments is shown in Fig. 1. It was designed and used for experiments on the flow of water through submerged orifices. To adapt the apparatus for use in the weir experiments the orifice plate between the galvanized iron head-tank and the weir-box was removed, giving a two-foot square opening through which the water entered the weir-box. The weir-box, or channel, was ten feet long, inside, 52 inches wide and 26½ inches deep below the crest of the weir. The water supply entered the galvanized iron head-tank through a three-inch pipe, the end terminating in a sheet iron cone perforated with 2-inch holes, which allowed the water to flow into the weir-box with very little disturbance. Two wooden gratings, composed of strips ½ in. by 1½ in. spaced ½ in. apart, were placed about two feet from the entrance to the weir-box, one grating having the strips vertical and the other horizontal and the distance between the gratings being about one inch. These gratings caused an appreciable loss of head and the openings through them discharged as submerged orifices. The velocity of flow in the weir-box was therefore uniform immediately below the gratings.

The various sizes of channels required were secured by means of three movable sections; a bottom and two sides. These were built of ⅞ in. matched pine flooring, 5 ft. long. They were held in place by being nailed to cleats on the bottom and sides of the box. In making the experiments the bottom was fixed at a definite distance below the crest of the weir, either 6 in., 12 in., 18 in., or 26½ in., and a set of runs made for each of a number of widths of channel. The various widths for the 12-inch weir are shown in Fig. 2. The channels were provided with flaring sheet metal entrances to prevent the formation of eddies.

The head on the weir was measured by means of a hook-gage, reading to 0.001 ft., in a still basin fastened to the side of the weir-box and communicating with the channel by a flexible tube.

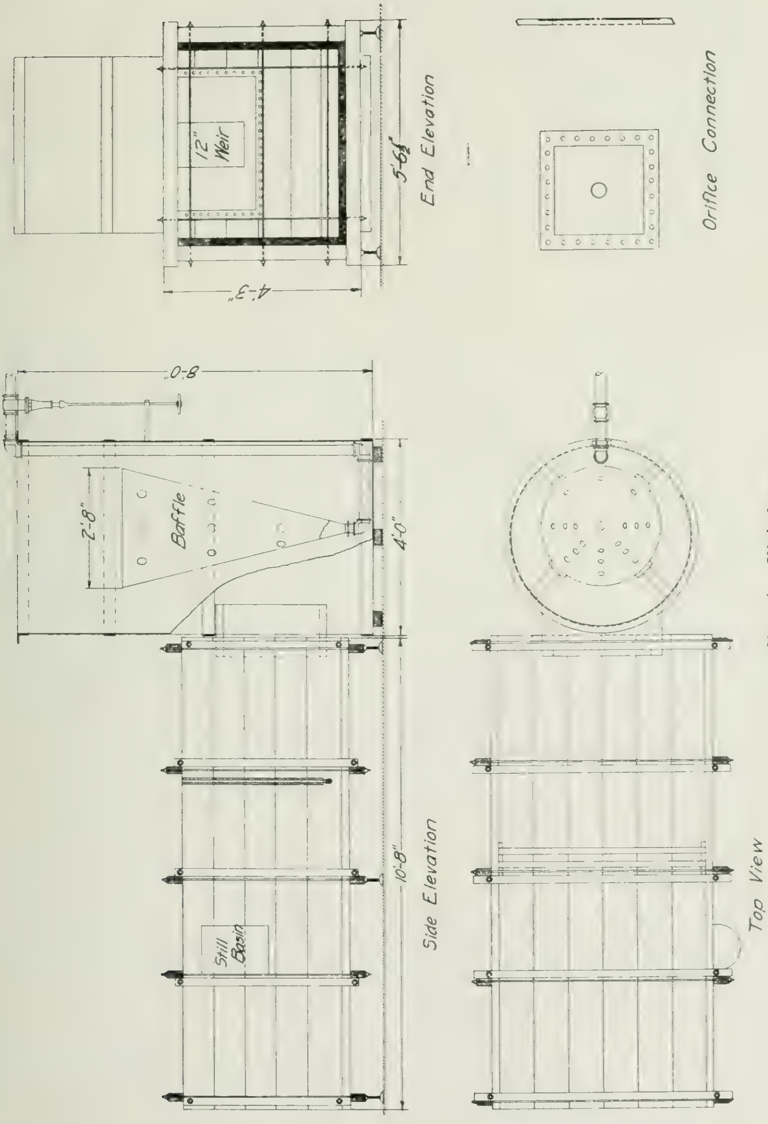

FIG. 1. Weir-box.

The piezometer opening in the side of the channel was a circular hole in a brass plate which was sunk flush with the surface of the wood. This opening was about 3½ ft. back from the weir.

The weirs were cut in ⅛ in. steel plate, with the edges beveled on the outside. The sides of the channel did not project beyond the weir, so the sheet of water flared laterally after passing the crest.

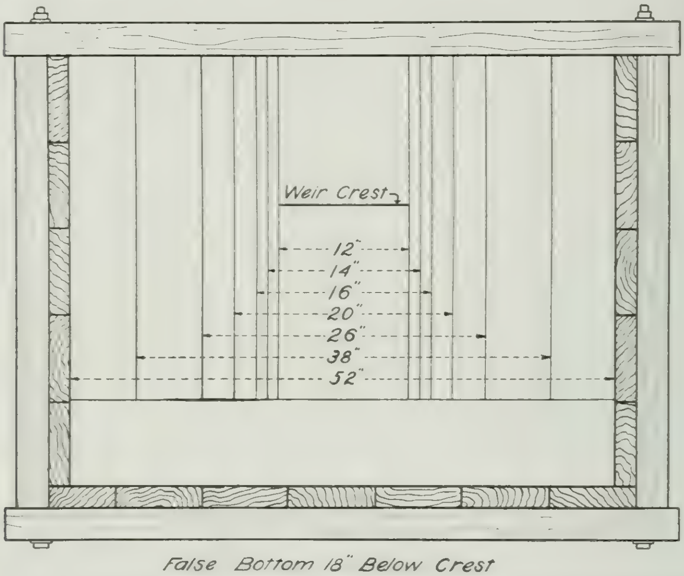

FIG. 2. Cross-section of Weir Box showing various Widths of Channel used in Experiments on the 12-in. Weir.

The quantity of water flowing over the weir was measured volumetrically in a calibrated measuring basin, which formed part of the basement of the hydraulic laboratory. Measuring tank D, shown in Fig. 3, was used in these experiments. It was about 10 ft. by 16 ft. in plan and 9 ft. high. Two sluice gates, about 2 ft. square, shown in the drawing, and one circular 12-inch gate are in the walls. The 2-foot gate leading into the lower waste channel was used for emptying the tank. It was possible to enter the waste channel and inspect this gate, while under pressure, for the detection of leaks. The other gates,

were not used during the experiments and were calked tight. The water entered the basement tank from the weir through an opening in the main floor on which the weir-box stood. Lying on the floor of the measuring tank is a perforated iron pipe, which is connected to a glass gage in the pump room by a pipe embedded in the concrete floor. The wooden scale attached to the gage was graduated into hundredths of a foot. By means of it it was possible to determine the elevation of the water surface in the tank.

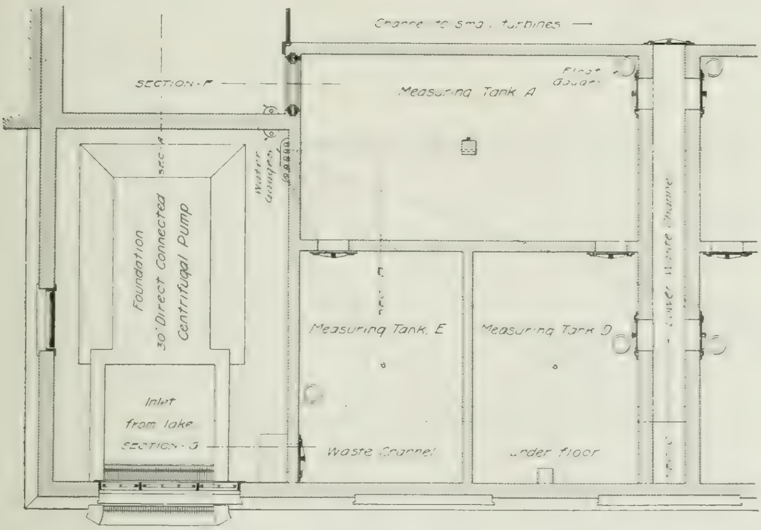

FIG. 3. Part of Basement Plan of Hydraulic Laboratory showing Measuring Basins.

METHOD OF EXPERIMENTATION

In the following discussion the term *run* will be used to designate such a combination of readings, made under practically constant conditions, as form a complete unit in the result; the term *series* is used to designate those runs made with varying rates of flow but with constant conditions of apparatus.

GAGING THE MEASURING BASIN

The measuring basin was gaged by introducing known quantities of water and noting the elevation of the water in the basin gage after each addition. This operation, though simple, proved difficult of accurate accomplishment, the first gagings by different parties apparently differing by several per cent. The following method was finally adopted and gave results which checked satisfactorily. All visible leaks in the basin were stopped and the sluice gate adjusted so that it closed water-tight. It was found that these gates buckled sufficiently to cause appreciable leakage when under the stress required to force them tightly shut, so after shutting the gate the stem was relieved of strain, and the gate was examined for leaks after the tank was partly full of water. Two galvanized iron tanks on platform scales were used for measuring the water. The scales were put into position for use and afterwards calibrated by standardized weights. Water was then run into the tanks, filling them alternately, the temperature and weight of each tankful of water, the elevation of the water in the measuring basin, and the temperature of the basin gage being observed and recorded. A sample sheet of data is shown in Table 1. The tank gage was read to thousandths of a foot, one one-thousandth being the rise occasioned by about ten pounds of water.

The volume of each tankful of water weighed into the basin was calculated by dividing the observed weight by the density of pure water at the observed temperature; careful observations having shown that the density of the lake water used was the same as that of pure water, out to the fifth decimal place.

Owing to the projection of gates and other irregularities in the walls of the basin, its horizontal cross-sectional area varied appreciably at different heights, so sufficient accuracy could not be secured by using the average area. The volume of water in the basin at various depths could not be conveniently plotted, against gage heights, to a sufficiently large scale to show the discrepancies in the various calibrations, so the following method was adopted. Since the smallest horizontal area of the basin

was a little above 160 square feet the volume of contained water above the zero of the tank gage is given by

$$V=160\ G+X$$

in which G is the tank gage reading and X may be called the excess.

It was not possible to read lower than about 0.3 ft. on the gage scale, owing to its being placed too low, so the absolute quantity of water in the basin above the zero of the gage could not be determined by the above formula, but the excess volume, x, between any two gage heights, g_a and g_b is given by the formula

$$v=160(g_b-g_a)+x$$

in which v is the actual volume of the tank between gage heights g_a and g_b. In gaging the basin the elementary excess for each tankful, of about 1800 pounds, of water was calculated, and these were summed as shown in the column headed cumulative excess, which shows the total excess between any observed gage height and the initial gage height of that particular gaging.

In order to compare the results of different gagings of the basin, it was necessary to determine what the excess would have been had all the initial gage readings been the same. This was done by dividing the data of each gaging into two groups of observations and determining the average gage reading and the average value of the elementary excesses for each group and also for the entire gaging. These values when plotted on cross section paper will lie on a straight line, if the calculation is correctly done. The intercept of this line on the X axis was added to or subtracted from the values of the cumulative excess and the resulting values were plotted as abscissas, against gage heights as ordinates. A straight line averaging the points passed through the origin and the results of all the gagings were thus put on a comparative basis. As may be seen in Fig. 4, the results of a number of gagings when reduced in this manner agree within one-twentieth of one per cent. If the plotted values deviate much from a straight line this method will not give satisfactory results

TABLE No. 1

THE UNIVERSITY OF WISCONSIN, HYDRAULIC LABORATORY

Experiment on Basin Calibration of Basin Number 4
Data by Curwen & Broker
Date of Experiment July 14, 1910
General Data: Tank No. 1 on Scale No. 852683
Tank No. 2 on Scale No. 799141
Initial Reading of Gage 0.671 Feet.
Temperature 23.7° C. Intercept 0.20

Computed by Curwen & Broker

Averages:

h h	Cumulative Excess
2.077	10.579
4.095	21.224
6.114	31.870

Tank No. 1. Initial Weight.	Tank No. 1. Gross Weight.	Tank No. 1. Net Weight.	Tank No. 1. Corrected Weight.	Tank No. 2. Initial Weight.	Tank No. 2. Gross Weight.	Tank No. 2. Net Weight.	Tank No. 2. Corrected Weight.	Temperature of water in degrees Cent.	Gage height in feet.	Temperature of gage in degrees C	Sum of corrected weights of tanks Nos. 1 and 2.	Cubic feet.	Total cubic feet	$h-h_o$	$160(h-h_o)$	Cumulative excess.	Corrected excess.
213.0	1658.5	1445.5	1441.7	213.0	1345.75	1132.75	1129.7	25.5	0.908	23.4	2571.4	41.319	41.319	0.237	37.92	3.40	3 60
217.25	1675.5	1458.25	1454.4	221.5	1357.0	1136.5	1133.3	25.3	1.160	23.4	2587.7	41.580	82.899	0.489	78.24	4.70	4.90
271.0	1699.75	1428.75	1434.8	240.25	1515.75	1275.5	1271.9	25.3	1.435	23.3	2696.7	43.334	126.233	0.764	122.24	3.99	4.19
279.0	1660.5	1381.5	1377.7	240.5	1416.0	1175.5	1172.2	25.2	1.688	23.6	2553.2	41.025	167.253	1.017	162.72	4.54	4.74
266.25	1661.75	1395.5	1391.7	223.0	1338.0	1115.0	1111.9	25.2	1.934	23.8	2506.7	40.278	207.536	1.263	202.08	5.46	5.66
235.5	1683.0	1448.5	1444.6	2202.5	1381.0	1160.75	1157.6	25.2	2.186	23.7	2602.2	41.812	249.348	1.515	242.40	6.95	7.15
229.5	1637.5	1408.0	1404.3	224.5	1427.0	1202.5	1199.2	25.2	2.441	24.0	2603.5	41.832	291.180	1.770	283.20	7.98	8.18
219.25	1721.5	1502.2	1498.3	226.5	1386.5	1160.0	1156.8	25.1	2.700	24.2	2655.1	42.663	333.843	2.029	324.64	99.20	9.40
221.5	1655.5	1434.0	1430.2	225.25	1344.25	1119.0	1115.9	25.2	2.936	24.3	2546.1	40.913	374.756	2.265	362.40	12.36	12.56
278.25	1618.5	1340.25	1336.6	280 5	1332 0	1051.5	1048.5	25.1	3.177	24.6	2385.1	38.327	413.083	2.506	401.46	11.62	11.82
215.5	1687.25	1471.75	1467.8	238.75	1340.25	1101.5	1098.4	25.1	3.421	24.2	2566.2	41.235	454.318	2.750	440.00	14.32	14.52
245.5	1670.0	1424.5	1420.7	226.5	1342.5	1116.0	1112.9	25.0	....	24.0	2533.6	40.711	495.029	....	....	....	......
225.2	1691.5	1466.2	1462.4	226.25	1382.75	1156.5	1153.3	25.0	3.931	24.3	2615.7	42.031	537.060	3.260	521.60	15.46	15.66

220.0	1729.5	1509.5	1505.5	221.0	1381.25	1160.25	1157.0	25.0	4.185	24.4	2662.5	42.787	579.847	3.514	562.24	17.61	17.81
240.0	1732.75	1492.75	1488.7	226.75	1365.0	1138.75	1135.1	25.0	4.433	24.4	2623.8	42.161	622.008	5.762	601.92	20.10	20.30
281.0	1714.0	1433.0	1429.1	256.5	1346.0	1089.5	1086.4	24.9	4.680	24.4	2515.5	40.420	662.428	4.009	641.14	20.99	21.19
273.0	1738.25	1465.25	1461.3	273.0	1444.25	1171.25	1177.9	25.0	4.940	24.2	2639.2	42.411	704.839	4.261	681.76	23.08	23.28
255.5	1743.5	1488.0	1483.9	225.75	1431.0	1205.25	1201.9	24.9	5.206	24.4	2685.8	43.175	748.014	4.535	725.60	22.41	22.61
244.5	1729.25	1514.75	1510.8	220.5	1436.25	1215.7	1212.4	24.9	5.469	24.4	2723.2	43.756	791.770	4.798	767.68	24.09	24.29
244.75	1685.75	1471.0	1467.1	232.25	1443.0	1210.75	1207.5	24.8	5.731	24.2	2674.6	42.981	834.751	5.060	809.60	25.15	25.35
244.25	1817.5	1603.25	1599.0	223.0	1426.25	1203.25	1200.0	24.8	6.004	24.4	2799.0	44.977	879.728	5.333	853.28	26.45	26.65
249.5	1770.0	1550.5	1546.4	220.0	1370.5	1150.5	1147.3	24.8	6.257	24.3	2693.7	43.285	923.013	5.586	893.76	29.25	29.45
247.25	1739.75	1492.5	1488.4	268.0	1481.5	1213.5	1210.0	24.7	6.523	24.2	2698.4	43.361	966.374	5.852	936.32	30.05	30.25
225.25	1747.0	1521.75	1517.6	264.0	1488.0	1224.0	1220.5	24.6	6.795	24.3	2738.4	44.013	1010.387	6.124	979.84	30.56	30.76
216.25	1808.25	1592.0	1587.8	223.25	1374.0	1150.75	1147.6	24.6	7.052	24.5	2735.4	43.957	1054.344	6.381	1020.96	33.36	33.56
291.0	1767.0	1473.0	1468.9	231.25	1400.0	1168.75	1165.5	24.6	7.318	24.4	2634.4	42.331	1096.675	6.647	1063.52	33.16	33.36
247.0	1769.75	1552.75	1548.7	220.0	1422.75	1202.75	1199.5	24.6	7.573	24.2	2748.2	44.160	1140.835	6.902	1104.32	36.52	36.72
293.0	1777.5	1484.5	1480.4	243.5	1417.25	1173.75	1170.5	24.5	7.837	24.3	2650.9	42.598	1183.433	7.166	1146.56	36.87	37.07
225.0	1822.7	1597.75	1593.5	230.0	1449.0	1189.0	1185.8	24.6	8.096	24.5	2779.3	44.659	1228.092	7.425	1188.00	40.09	40.29
297.25	1776.25	1479.0	1474.9	225.75	1441.5	1215.75	1212.4	24.5	8.356	24.3	2687.3	43.185	1271.277	7.685	1229.60	41.68	41.88
238.75	1802.0	1563.2	1559.1	227.25	1528.0	1300.75	1297.1	24.5	8.620	24.2	2856.2	45.888	1317.165	7.949	1271.84	45.33	45.53

unless the gagings cover practically the same range of elevations in the basin.

To compute the quantity of water, V, run into the basin, between gage heights G_1 and G_2 the following formula is used:

$$V = 160(G_2 - G_1) + (X_2 - X_1)$$

In this formula X_2 and X_1 are the corrected cumulative excess

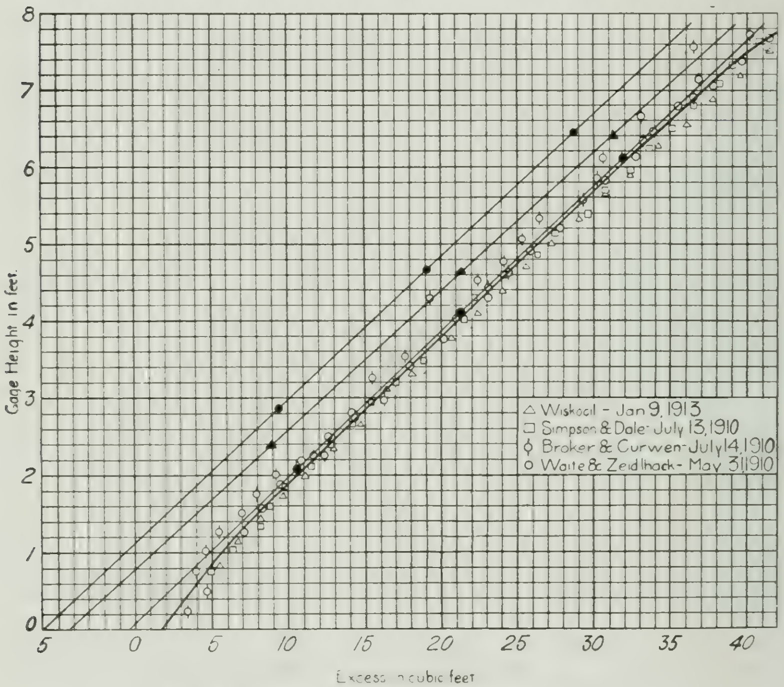

FIG. 4. Gaging Curves of Measuring Basin D.

for gage heights G_2 and G_1 respectively, and are taken from a curve similar to Fig. 4 plotted to a large scale.

Calibrating the Weirs

The data observed during the experiments were time, elevation of water in measuring tank, and head on the crest of the weir. The elevation of the water in the measuring tank was observed

before the beginning of a run and at the end after the oscillations of the gage had ceased. Between runs the water from the weir was diverted from the measuring tank into the waste channel. The head on the weir was read at equal intervals of time, and from five to sixteen times during a run, the runs being longer and the head being read more frequently in the work on the 12-in. weir than in the other work. The results from the 12-in. weir are accordingly more reliable and more consistent than the others.

The pressure in the water supply mains was quite constant, so the flow was very uniform, as may be seen in Table 2, which shows a representative run. A summary of all the data is given in Table 3.

TABLE 2

12 inch Weir

Series 2. Run No. 5. Height of Crest 12″. Width of Channel 52″

Hook Gage Readings	Duration of Run in Seconds	Tank Gage Readings	Total Discharge in Cu. ft.	Discharge in Second Feet	Coefficient "C"
.1525	....	0.269	....	....	....
.1525	....	....	....	....	....
.1525	....	....	....	....	....
.1525	....	....	....	....	....
.1525	....	....	....	....	....
.1530	....	....	....	....	....
.1530	....	....	....	....	....
.1525	....	....	....	....	....
.1520	....	....	....	....	....
.1520	....	....	....	....	....
.1520	....	....	....	....	....
.1520	....	....	....	....	....
.1525	....	....	....	....	....
.1525	....	....	....	....	....
.1525	....	....	....	....	....
.1525	....	2.424	....	....	....
.1524	1800.00	2.155	355.02	0.1973	3.316

DISCUSSION OF RESULTS

A sharp crested weir in the side of a deep, wide reservoir would have perfect contractions and would be free from the effects of velocity of approach. Under such conditions the discharge under low heads would be in accordance with the formula

$$q = m\ l\ h^n$$

in which m and n are nearly constant, the latter being slightly less than 1.5. The variability of m and n under the above conditions is due to the viscosity of the liquid flowing over the weir.

When the flow of the weir takes place through a restricted channel the variation in m and n increases due to the effects of velocity of approach. The losses due to viscosity are increased due to the greater relative motion of the stream lines, and also to the eddy motion which occurs under these conditions. The measured head, H, does not include the velocity head which becomes an appreciable quantity under some conditions. The contraction of the sheet of falling water is modified by the direction of the currents approaching the weir. Some of these effects tend to counter-balance the others but the net effect is to increase both m and n as the size of the channel decreases. Even when the width of channel and height of crest are constant the values of m and n increase as the head increases, so that when the logarithm of H and the logarithm of q are plotted the points fall on a curved line. The curvature of this line might be caused by an error in the zero reading of the hook gage, the location of the piezometer opening too near the weir in the drop down curve, the velocity of approach, and imperfect contractions, and would tend to be reversed by the effects of viscosity and eddy losses. In the present experiments the zero hook gage reading was frequently checked and the hook gage was located 3½ feet back of the crest, so the drop in the water surface at that point was quite small, under the low heads experimented with, and the curvature of the logarithmic curves, which was quite noticeable, was largely due to the other causes mentioned.

On account of the variation in m and n a simple exponential formula will not express the relation between discharge and head with sufficient accuracy over a large range of heads. It was therefore thought best, for the purposes of the present study, to use the formula

$$q = cl\,H^{\frac{3}{2}}$$

in which c is a variable.

The actual rate of discharge of each run was, therefore, divided by l H, giving the coefficient of discharge c. This coefficient includes the corrections for velocity of approach, fluid friction, imperfect contraction, and all other effects due to the differences between actual and ideal conditions. The values of c have been plotted as ordinates against the observed head as abscissas in Figs. 5, 6, 7, 8, 9, and 10. Curves have been drawn, for each series of runs, which average the points quite well in most cases. There are some series in which points lie some distance from the curves, which are located by interpolition, and which indicate serious errors in those series.

For comparison with the above there have been plotted in Fig. 11, the results of the experiments by W. G. Steward and J. S. Longwell, published in the *Proceedings of the American Society of Civil Engineers,* Vol. 39, Feb. 1913, and in Fig. 12, the results of experiments by M. Bazin published in the *Proceedings of the Engineer's Club of Philadelphia,* Vol. 7, Jan. 1890.

Effect of Length of Weir

The velocity of water approaching a suppressed weir is greatest at the middle of the channel and less near the sides, due to the resistance caused by the side walls of the approach channel. For any given velocity of flow the retarding effect of the side walls would be appreciable a definite distance from the ends of the weir and as the length of the weir is increased, other conditions remaining constant, a smaller proportion of the total flow would be disturbed by the side walls. From this it would seem reasonable to suppose that, as the length of the suppressed weir is increased, the coefficient of discharge would increase.

As may be seen in Table 4, the values of which have been taken from Figs. 5–10, there seems to be a tendency for the coefficient to increase with the length of the weir. The data are, however, somewhat contradictory and are too meager to base any general conclusion upon, especially in view of the fact that the tables of Hamilton Smith, Jr. indicate that the co-

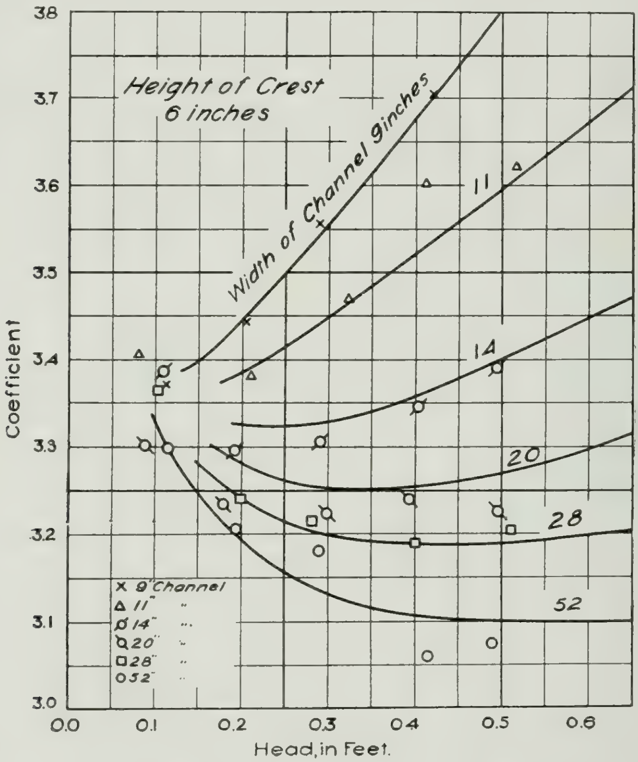

FIG. 5. Relation of Coefficient to Head. 9-inch Weir.

efficient should increase as the length decreases, and the experiments of M. Bazin did not show any appreciable effect of length on the coefficient of suppressed weirs. It should be noted here that in the Wisconsin experiments the side walls of the channel were not prolonged beyond the crest of the weirs as they were in the other experiments referred to. According to Hamilton Smith, Jr., the coefficients of a suppressed weir having a per-

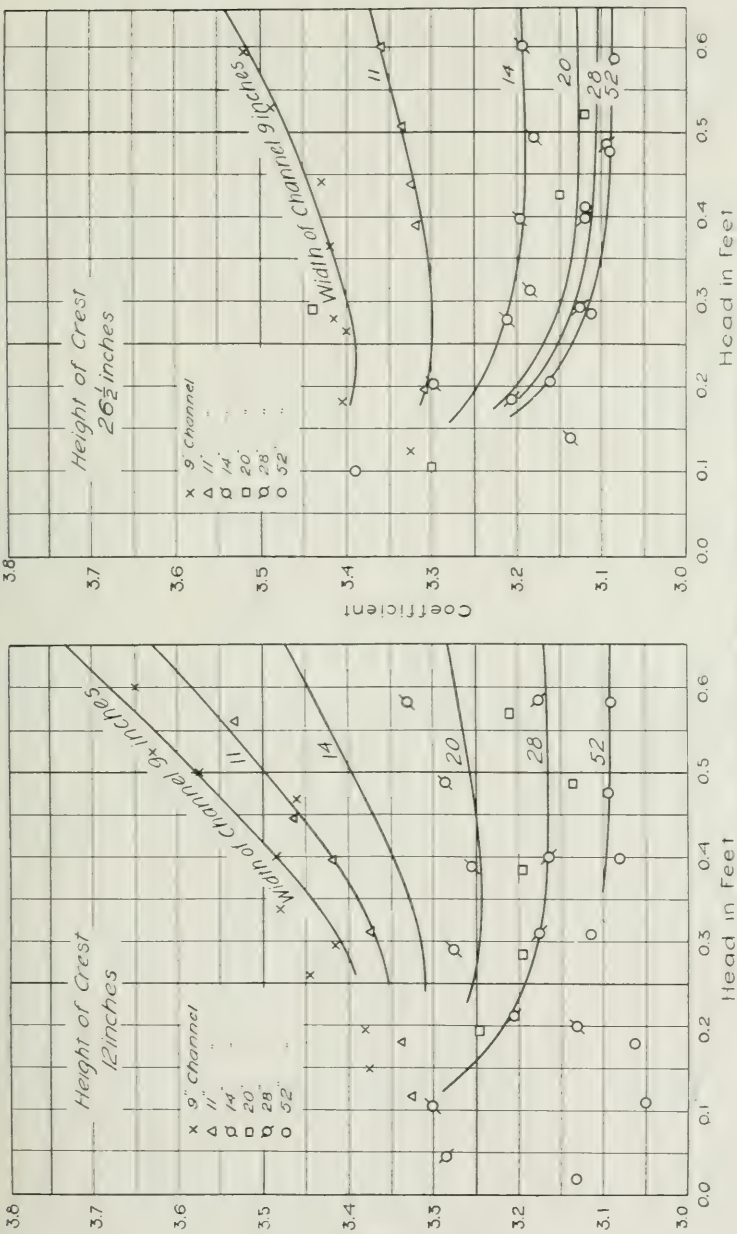

FIG. 6. Relation of Coefficient to Head. 9 inch Weir.

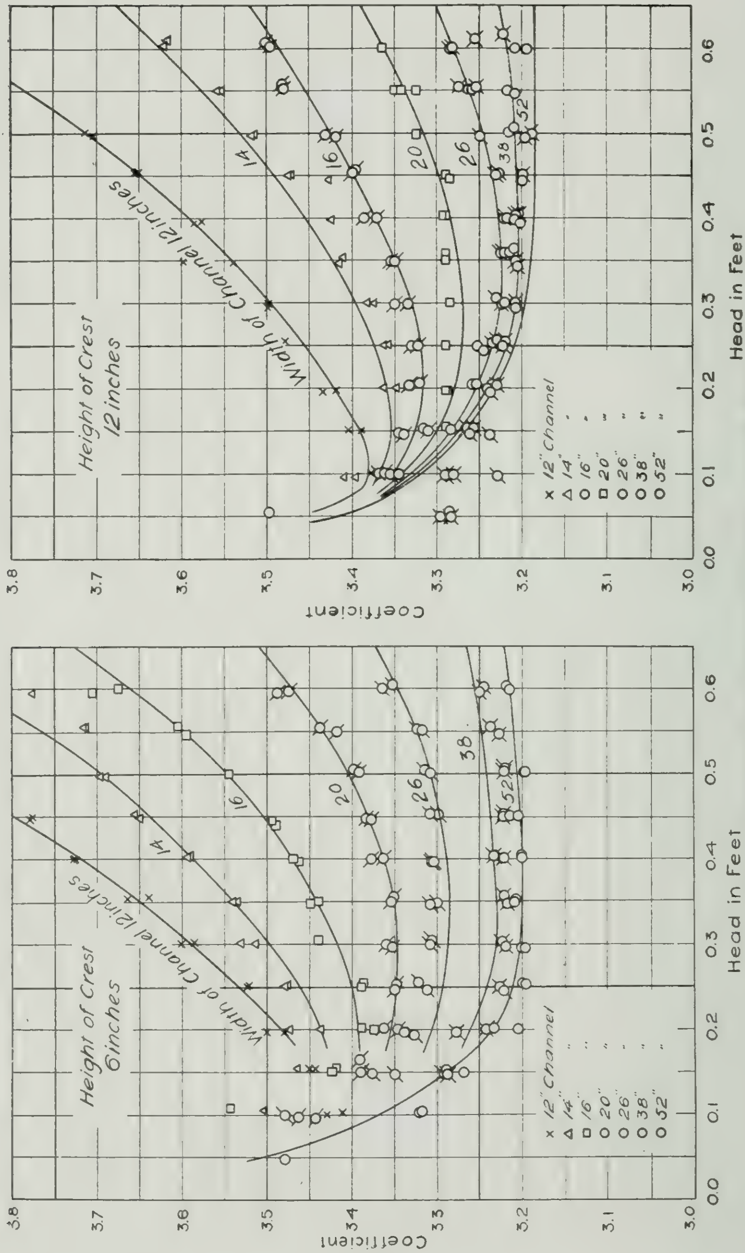

FIG. 7. Relation of Coefficient to Head. 12-inch Weir.

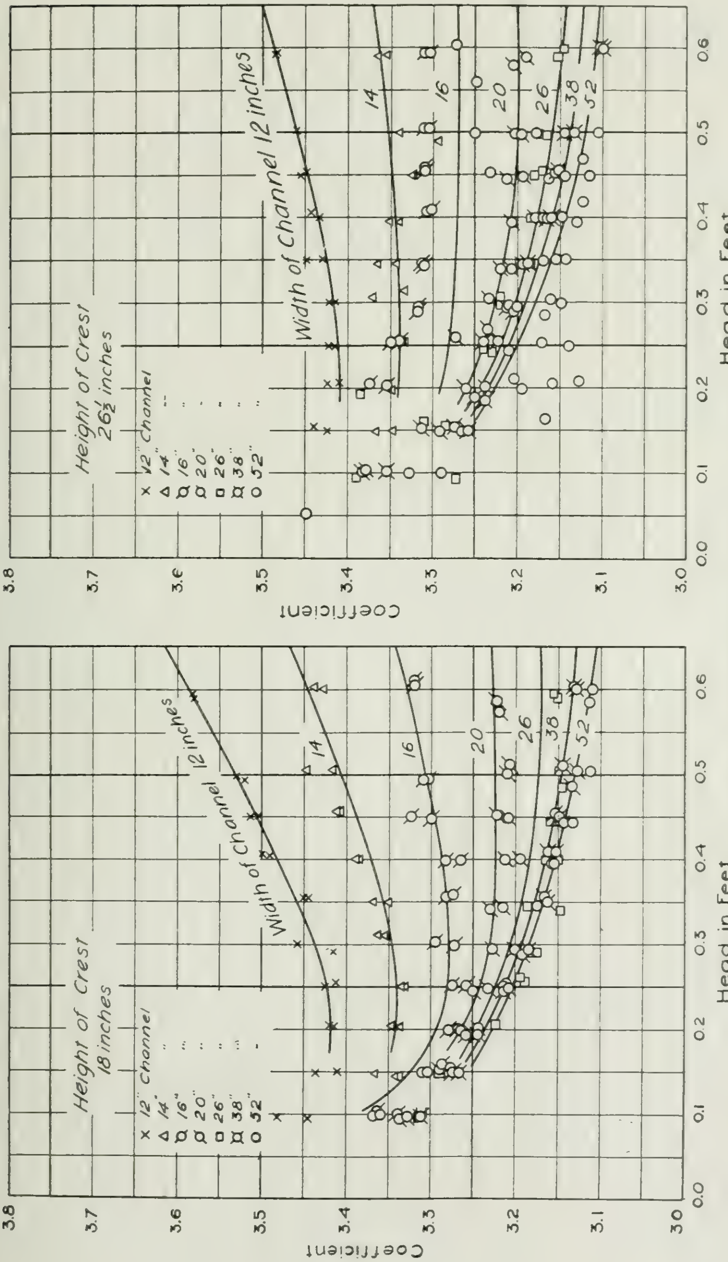

FIG. 8. Relation of Coefficient to Head. 12-inch Weir.

fectly free discharge are from about $\frac{1}{4}$ of 1 per cent. lower than those of a weir with the sides prolonged past the crest so as to prevent the lateral flaring of the sheet.

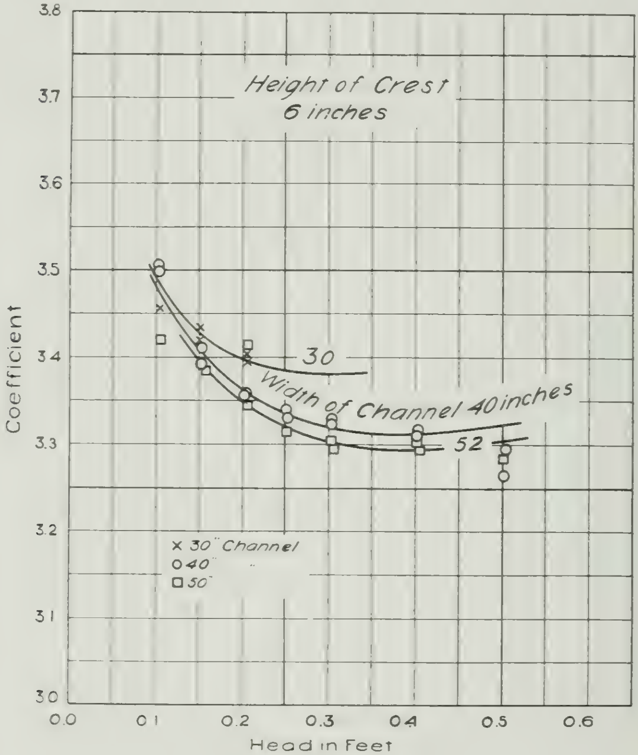

FIG. 9. Relation of Coefficient to Head. 18-inch Weir.

These experiments indicate that the coefficients of weirs having end contractions increase as the length of weir increases, as may be seen in Table 5, which has been made up by reading from Figs. 5–10, the values of the coefficients for channels having the side-walls a distance H from the ends of the weirs. This is in agreement with the tables of Hamilton Smith, Jr. and the results of the experiments of M. Bazin and is accounted for by the fact that the effect of the end contractions, which extends for an indefinite distance from the ends of the weir, is dependent on the head, but is independent of the length of the weir. The av-

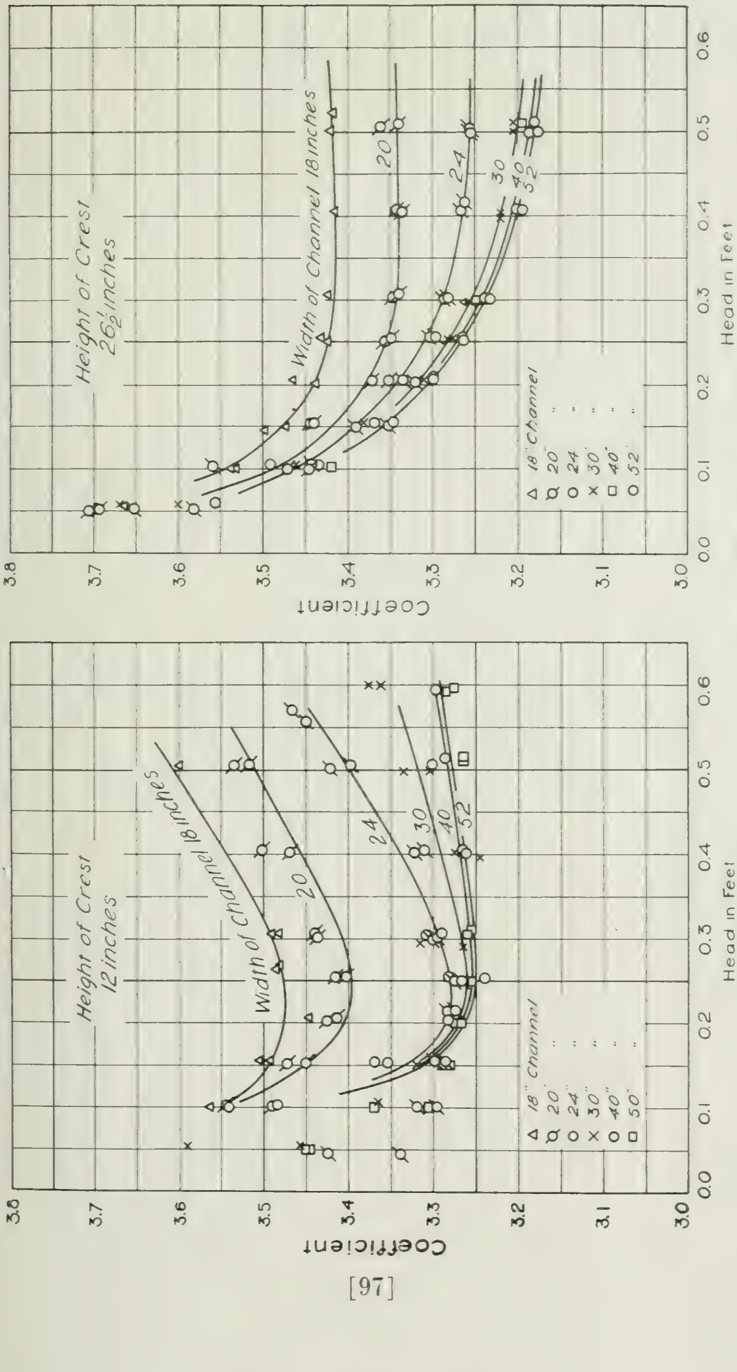

FIG. 10. Relation of Coefficient to Head. 18 inch Weir.

erage discharge per length of weir should, therefore, increase with the length. The relation seems to be reversed for high heads in the experiments of Steward and Longwell.

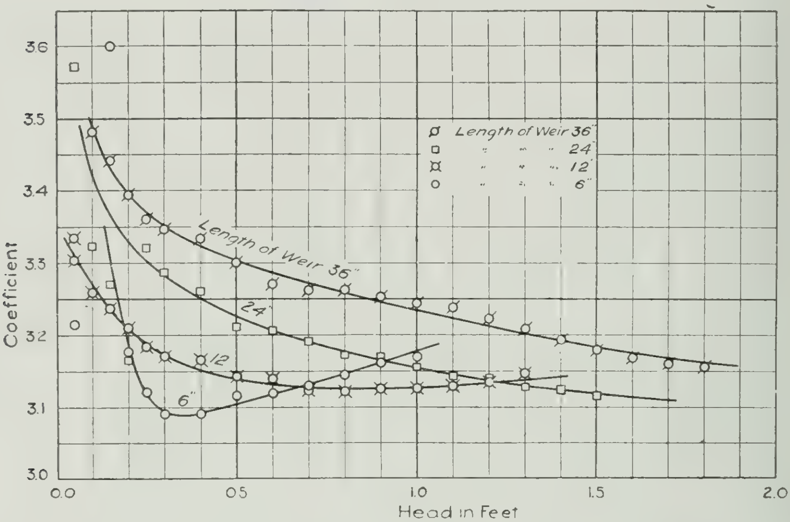

FIG. 11. Results of Experiments made by Steward and Longwell.

TABLE 4

Relation of Coefficient to Length of Weir

End Contractions Suppressed

Head = 0.3 foot

Height of Crest	9 inch weir	12 inch weir	18 inch weir
6 inch	35.6	35.8	....
12 inch	34.1	35.0	34.9
26½ inch	34.0	34.2	34.2

Head = 0.5 foot

6 inch	38.1	38.8	....
12 inch	35.8	37.2	36.1
26½ inch	34.7	34.6	34.2

Effect of Height of Crest

The effect of the height of the crest on the coefficient is shown in Fig. 13, which has been plotted from data taken from the curves for the 12-inch weir. In these figures, it may be seen that under very low heads the rate of decrease in the coefficient

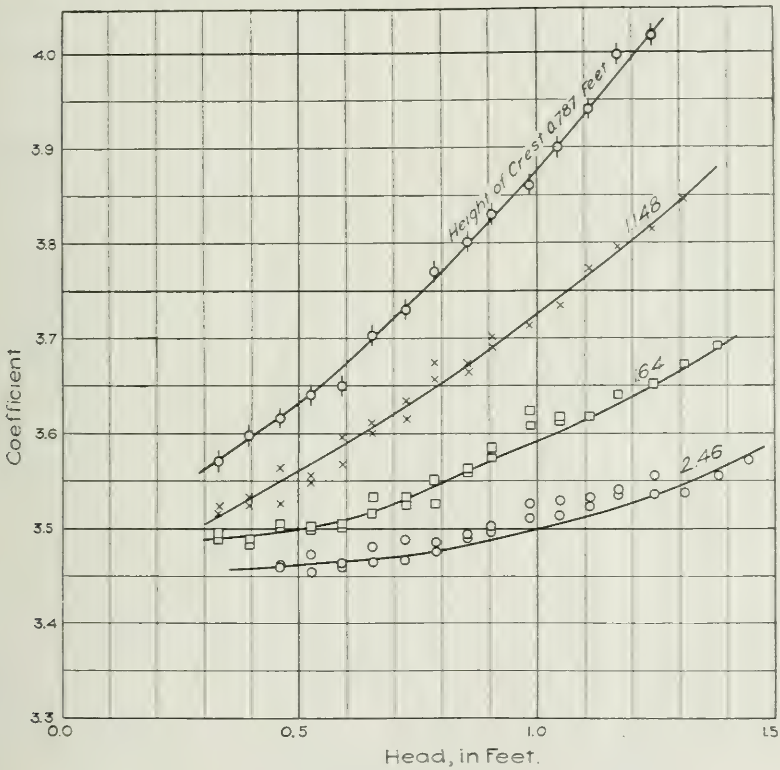

Fig. 12. Results of Experiments made by M. Bazin.

is greatest when the height of crest is less than one foot, while under the higher heads experimented with the rate of decrease is greatest between 12-and 18-in. height of crest.

The change in the coefficient, with variations in the height of the crest, is due to three causes: first, the difference between the measured head and the true head, due to velocity of approach and the drop-down curve; second, varying degrees of suppression of the contraction of the sheet; third, losses of head due to

friction, between the hook gage and the weir. In computing the values of the coefficient the observed or apparent head has been used and since this is always less than the true head the coefficient should increase as the velocity increases, other things being equal. The loss due to friction would tend to reduce the coefficient as the channel gets shallower, but this effect is very small and is usually negligible. The contraction of the sheet is due to the fact that those particles of water that approach the notch from regions below the crest or beyond the ends of the weir move in curved paths, the tangents to which, at all points, lie in the direction of the resultant of all the forces acting on the particle of water. When the bottom and sides of the channel are near the crest and ends of the weir the mass of water approaching the notch from a lateral direction is reduced and the mass approaching in a longitudinal direction has a greater velocity, so the resultant of the forces due to hydrostatic pressure and momentum acting on any particle near the crest or ends of the weir

TABLE 5

Effect of length of weir on coefficient

Side walls distant H from ends of weir

Head = 0.3 ft.

Height of Crest	9 inch weir Width of channel = 16.2 in.	12 inch weir Width of channel = 19.2 in.	18 inch weir Width of channel = 25.2 in.
6 inch	33.1	33.5	34.4
12 inch	32.8	32.8	32.9
26½ inch	31.9	32.3	32.8

Head = 0.5 ft.

Height of Crest	9 inch weir Width of channel = 21 in.	12 inch weir Width of channel = 24 in.	18 inch weir Width of channel = 30 in.
6 inch	33.0	33.3	34.1
12 inch	32.4	32.7	33.1
26½ inch	31.2	31.7	32.0

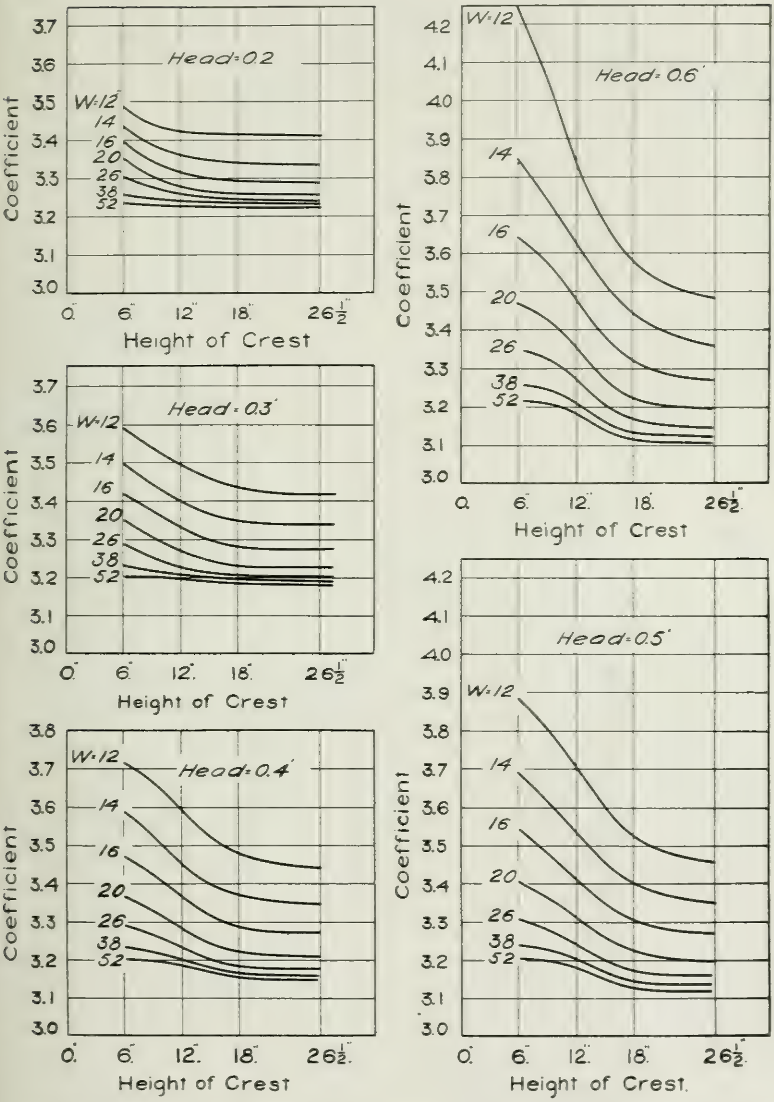

Fig. 13. Showing the Effect of the Height of Crest on the Coefficient of Discharge.

has a direction more nearly normal to the plane of the weir, and the contraction is thereby more or less suppressed.

On account of the nonuniform distribution of velocities in the approach channel, and the uncertainty as to the effect such a distribution has on the velocity head, no attempt has been made, in these experiments, to separate the effects of velocity of approach, and of friction, from the effects of suppression of contraction. The first and last are the more important and operate in the same direction.

It may be noted in the curves that under low heads, up to about 0.3 ft., a height of crest of 18 inches was apparently sufficient to practically eliminate all error due to the above causes, for all widths of channel. For higher heads and suppressed and nearly suppressed weirs a height of crest of even 26½ inches was not enough to entirely eliminate all error. A height of crest of 5H or more would doubtless have been required in the case of complete suppression of the end contractions to reduce the variation of the coefficient to less than one per cent from the minimum value.

Effect of Width of Channel

The effect of varying the width of the channel is shown in Fig. 14 which has been taken from the curves for the 12-inch weir. In this figure it may be seen that the coefficient diminishes rapidly with slight increases in the width of the channel. In the case of the weir having a height of crest of 26½ inches the coefficient apparently reaches a value within one per cent of its minimum with a width of channel of 36 inches. In this case the sides of the channel are 2H from the ends of the weir.

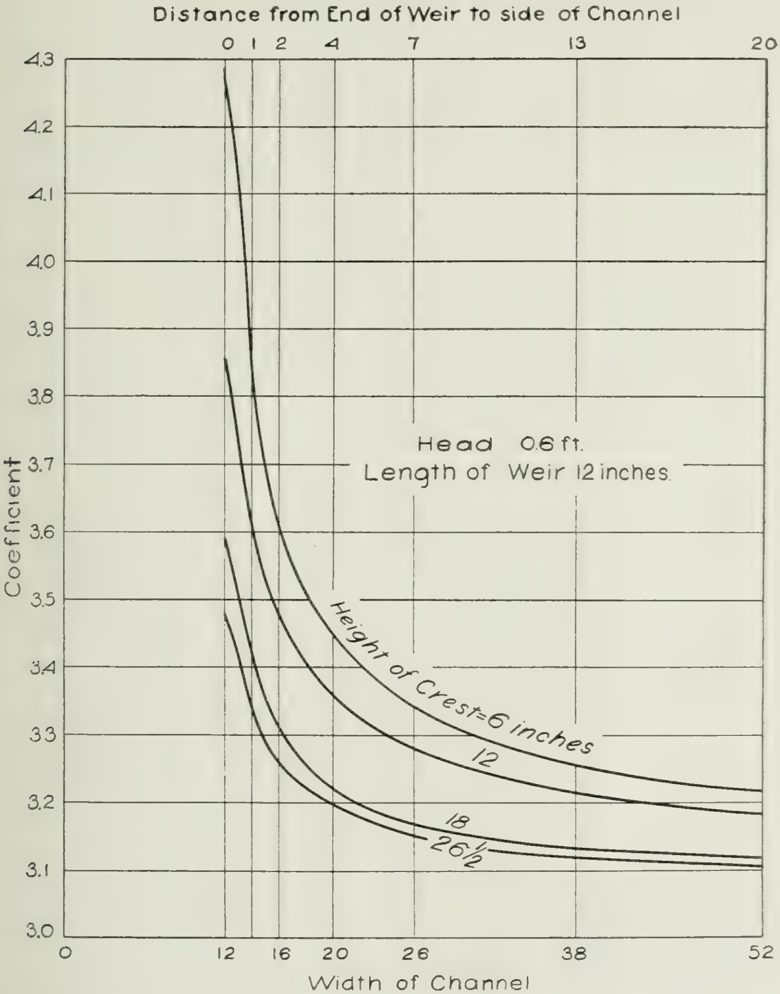

Fig. 14. Showing the Effect of Varying the Width of Channel.

TABLE NO. 3
Summary of Data
9 inch Weir
Series 1. Height of Crest 6″ Width of Channel 9″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	900	16	.2045	.2388	3.443
2	900	16	.2872	.4156	3.558
3	900	16	.1122	.0905	3.371
4	600	11	.5057	1.0307	3.824
5	600	11	.4197	.7563	3.709
6	600	11	.6128	.3857	3.852

9 inch Weir
Series 2. Height of Crest 6″ Width of Channel 11″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	900	16	.0817	.0596	3.4030
2	900	16	.2083	.2409	.3802
3	900	16	.3219	.4752	.4692
4	600	11	.6267	1.3670	.6760
5	600	11	.4104	.6950	.6039
6	600	11	.5155	1.0053	.6232

9 inch Weir
Series 3. Heighth of Crest 6″ Width of Channel 14″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	900	16	.1100	.0926	3.3847
2	600	11	.6339	1.3167	3.4782
3	900	16	.1924	.2087	3.2873
4	900	16	.2910	.3894	3.3070
5	600	11	.4028	.0418	3.3470
6	600	11	.4940	.8832	3.3918

TABLE No. 3
Summary of Data
9 inch Weir
Series 4. Height of Crest 6" Width of Channel 20"

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	900	16	.1785	.1828	3.2325
2	600	11	.4951	.8432	3.2260
3	900	16	.0880	.0647	3.3014
4	900	16	.2983	.3931	3.2267
5	600	11	.3929	.5982	3.2390
6	600	11	.6977	1.4350	3.2832

9 inch Weir
Series 5. Height of Crest 6" Width of Channel 28"

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet.	Coefficient
1	900	16	.1025	.0828	3.3648
2	900	16	.1992	.2162	3.3242
3	600	11	.3997	.6043	3.1883
4	600	11	.5091	.8722	3.2028
5	900	16	.2827	.3625	3.2157
6	600	11	.6464	1.2555	3.2213

9 inch Weir
Series 6. Height of Crest 6" Width of Channel 52"

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	900	16	.1143	.0956	3.2980
2	900	16	.2880	.3687	3.1810
3	900	16	.1936	.2044	3.2062
4	600	11	.4887	.7875	3.0736
5	600	11	.4142	.6102	3.0580
6	600	11	.6685	1.2840	3.1322

TABLE NO. 3
Summary of Data
9 inch Weir

Series 7. Height of Crest 12″ Width of Channel 11″

Run No.	Duration of Run in Seconds.	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	900	16	.0939	.0692	3.2062
2	900	16	.2947	.4100	3.4168
3	900	16	.1946	.2177	3.3822
4	600	11	.4995	.8463	3.5726
5	600	11	.3983	.6565	3.4820
6	420	8	.5987	1.2690	3.6506
7	900	16	.1495	.1464	3.3768
8	900	16	.3359	.5080	3.4790
9	900	16	.2569	.3362	3.4432
10	600	11	.5229	1.0293	3.0298
11	600	11	.4673	.8290	3.4600

Series 8. Height of Crest 12″ Width of Channel 9″

1	900	16	.1798	.1908	3.3360
2	600	11	.4462	.7743	3.4640
3	600	11	.3975	.6420	3.4160
4	600	11	.5590	1.1066	3.5300
5	900	16	.1145	.0966	3.3246
6	900	16	.3113	.4390	3.3705

Series 9. Height of Crest 12″ Width of Channel 14″

1	900	16	.1058	.0852	3.3006
2	900	16	.2916	.3868	3.2767
3	900	16	.2015	.2213	3.1310
4	600	11	.4872	.8377	3.2845
5	660	11	.3892	.5926	3.2545
6	600	11	.5821	1.1092	3.3295
7	960	18	.0895	.0756	3.5102

TABLE No. 3

Summary of Data

9 inch Weir

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
Series 10 Height of Crest 12″ Width of Channel 20″					
1	900	16	.0894	.0680	3.3916
2	900	16	.2918	.3774	3.1930
3	900	16	.1949	.2094	3.2459
4	600	11	.3852	.5727	3.1958
5	600	11	.4873	.8166	3.1358
6	600	11	.5669	1.0278	3.2110
Series 11 Height of Crest 12 Width of Channel 28					
1	900	16	.1119	.0927	3.3024
2	900	16	.3107	.4123	3.1743
3	900	16	.2129	.2361	3.2053
4	600	11	.4871	.8087	3.7652
5	660	11	.4002	.6007	3.1632
6	660	11	.5844	1.0644	3.1762
Series 12. Height of Crest 12″ Width of Channel 52″					
1	900	16	.1101	.0836	3.0511
2	900	16	.1814	.1774	3.0628
3	900	16	.3087	.4006	3.1138
4	600	11	.3979	.5800	3.0814
5	600	11	.5825	1.0303	3.0900
6	600	11	.4725	.7603	3.0947

TABLE No. 3
Summary of Data
9 inch Weir
Series 13. Height of Crest 26½" Width of Channel 9"

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1,200	21	.0331	.0156	3.0922
2	900	16	.0863	.0696	3.8390
3	900	16	.1816	.1977	3.4061
4	900	16	.2646	.3473	3.4022
5	900	16	.3627	.5597	3.4160
6	900	16	.5242	.9932	3.4895
7	600	11	.5939	1.2107	3.5270
8	720	13	.6769	1.4767	3.5365
9	600	11	.7288	1.6537	3.5450
10	900	16	.2765	.3726	3.4170
11	900	16	.1212	.1052	3.3245
12	600	11	.4398	.7502	3.4296
Series 14 Height of Crest 26½" Width of Channel 11"					
1	900	16	.09300	.0728	3.4219
2	900	16	.3027	.4101	3.2835
3	900	16	.1942	.2123	3.3080
4	900	16	.3876	.5998	3.3178
5	9C0	16	.5023	.8910	3.3370
6	600	11	.6811	1.4207	3.3700
7	600	11	.6006	1.1713	3.3610
8	900	16	.4386	.7244	3.3250
9	1200	21	.1268	.1117	3.2978
Series 18. Height of Crest 26½". Width of Channel 52"					
1	1,260	22	.0952	.0707	3.2090
2	900	16	.2850	.3551	3.1123
3	660	12	.4110	.6165	3.1198
4	360	7	.5871	1.0409	3.0850
5	900	16	.2051	.2202	3.1605
6	660	12	.4767	.7630	3.0910

Table No. 3
Summary of Data
9 inch Weir

Series 15. Height of Crest 26½″, Width of Channel 14″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	900	16	.0983	.0774	3.3480
2	900	16	.2025	.2204	3.2990
8	600	11	.2676	.3323	3.2110
4	600	11	.3968	.5992	3.1967
5	600	11	.4931	.8258	3.1800
6	600	11	.6012	1.1170	3.1943
7	600	11	.6994	1.4120	3.2167
8	600	11	.3122	.4165	3.1835
9	900	16	.1389	.1218	3.1372

Series 16. Height of Crest 26½″. Width of Channel 20′

1	600	11	.5206	.8787	3.1192
2	600	11	.5827	1.0402	3.1180
3	900	16	.1052	.8460	3.2978
4	600	11	.2885	.4000	3.4415
5	900	16	.1892	.1940	3.1428
6	600	11	.4250	.6546	3.1481

Series 17. Height of Crest 26½″. Width of Channel 28″

1	900	16	.3964	.5837	3.1185
2	900	16	.6331	1.1690	3.0941
3	600	11	.0958	.0732	3.2913
4	600	11	.2924	.3707	3.1260
5	900	16	.1837	.1886	3.2070
6	600	11	.4845	.7825	3.0935

TABLE No. 3

Summary of Data

12″ inch Weir

Series 1. Height of Crest 6″. Width of Channel 12″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1761	13	.3543	.7674	3.639
2	1831	13	.3521	.7651	3.663
3	1781	16	.3021	.5979	3.602
4	1772	16	.3061	.6071	3.586
5	1799	16	.0507	.0402	3.523
6	1773	16	.0503	.0396	3.511
7	1766	16	.1011	.1103	3.431
8	1753	16	.1019	.1110	3.412
9	1770	16	.1539	.2084	3.452
10	1767	16	.1549	.2099	3.444
11	1796	16	.1960	.3037	3.500
12	1767	16	.1960	.3018	3.478
13	1485	15	.2493	.4386	3.523
14	1403.5	16	.2495	.4392	3.525
15	1180	5	.6137	2.0370	4.238
16	1157	5	.6094	2.0300	4.268
17	983.5	6	.5553	1.6980	4.104
18	942.5	6	.5565	1.7015	4.099
19	776	7	.5008	1.3840	3.906
20	734	7	.5056	1.3850	3.880
21	632.5	9	.4459	1.1250	3.779
22	621.5	9	.4474	1.1300	3.776
23	527	11	.3981	.9370	3.730
24	529	11	.3996	.9424	3.731

TABLE No. 3
Summary of Data
12 inch Weir
Series 2. Height of Crest 6″. Width of Channel 14′

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1188	11	.4026	.9176	3.593
2	1171	11	.4025	.9170	3.591
3	1412	13	.3498	.7312	3.534
4	1420	13	.3500	.7330	3.540
5	1754	16	.3015	.5844	3.531
6	1778	16	.2995	.5760	3.514
7	595	6	.6015	1.7770	3.809
8	602	6	.5942	1.7290	3.775
9	704	7	.5526	1.5250	3.712
10	679.5	7	.5541	1.5320	3.714
11	834.5	8	.4952	1.2860	3.690
12	829	8	.4969	1.2950	3.697
13	963.5	6	.4529	1.1140	3.655
14	989	9	.4458	1.0860	3.649
15	1770	16	.0477	.0383	3.675
16	1761	16	.0487	.0394	3.667
17	1763	16	.1036	.1169	3.505
18	1770.5	16	.1023	.1137	3.475
19	1780	16	.1527	.2067	3.464
20	1773	16	.1543	.2089	3.446
21	1769	16	.1995	.3096	3.474
22	1791.5	16	.1996	.3065	3.437
23	1775	16	.2510	.4372	3.478
24	1773	16	.2514	.4379	3.476

TABLE No. 3

Summary of Data

12 inch Weir

Series 3. Height of Crest 6″. Width of Channel 16″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1638.5	15	.0983	.1129	3.662
2	1750.5	16	.1060	.1188	3.544
3	1831	16	.0461	.0361	3.646
4	1686	15	.0459	.0363	3.686
5	633	6	.5940	1.6965	3.706
6	638	6	.6001	1.7090	3.676
7	761 5	7	.5541	1.4860	3.603
8	750	7	.5471	1 4540	3.593
9	866.5	8	.5009	1.2570	3.546
10	872.5	8	.4989	1.2500	3.547
11	1059	9	.4473	1.0450	3 493
12	1071	10	.4438	1.0330	3.492
13	1317	12	.3992	.8772	3 478
14	1246	11	.3955	.8618	3.465
15	1624	14	.3459	.7018	3 451
16	1593	14	.3512	.7155	3 438
17	1804	16	.2997	.5964	3.634
18	1738	15	.3034	.5750	3 441
19	1735.5	15	.2498	.4235	3.391
20	1726	15	.2508	.4259	3.391
21	1816	16	.2022	.3084	3 392
22	1774	16	.2005	.3029	3.374
23	1735 5	15	.1537	.2062	3.422
24	1800	16	.1518	.2026	3.426

TABLE No. 3
Summary of Data
12 inch Weir
Series 4. Height of Crest 6″. Width of Channel 20″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1783	16	.2529	.4259	3.348
2	1739	15	.2457	.4083	3.352
3	1714	15	.2010	.3032	3.364
4	1868	16	.2006	.3009	3.349
5	1797	16	.1637	.2247	3.392
6	1750	16	.1507	.1984	3.392
7	1844	16	.0975	.1055	3.465
8	1755	16	.0965	.1036	3.395
9	1796	16	.0484	.0386	3.622
10	1701.5	15	.0484	.0389	3.641
11	1543.5	14	.3501	.6954	3.356
12	1587.5	14	.3508	.6972	3.355
13	1775.5	16	.2988	.5488	3.361
14	1752	16	.2967	.5421	3.355
15	671	7	.5946	1.6000	3.488
16	646	6	.5959	1.5985	3.475
17	769.5	7	.5538	1.4173	3.439
18	753	7	.5497	1.3930	3.419
19	878	8	.5034	1.2140	3.399
20	821	8	.5022	1.2076	3.393
21	1094	10	.4472	1,0120	3.385
22	1101	10	.4456	1.0050	3.379
23	1302.5	12	.3995	.8497	3.365
24	1252	11	.3989	.8511	3.379

TABLE No. 3
Summary of Data
12 inch Weir
Series 5. Height of Crest 6″. Width of Channel 26″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	685	7	.6007	1.5670	3.366
2	684	7	.6030	1.5699	3.353
3	787.5	8	.5522	1.3636	3.324
4	779	8	.5510	1.3574	3.319
5	916	9	.5043	1.1878	3.317
6	908	9	.5008	1.1735	3.311
7	1110	10	.4525	1.0047	3.300
8	1098	10	.4526	1.0080	3.310
9	1319	12	.3968	.8267	3.308
10	1298	12	.3970	.8272	3.307
11	1596.5	14	.3457	.6726	3.310
12	1609.5	14	.3479	.6775	3.302
13	1793	16	.2099	.5436	3.310
14	1794	16	.3032	.5524	3.309
15	1791	16	.2465	.4056	3.313
16	1774	16	.2550	.4280	3.323
17	1782	16	.1968	.2916	3.340
18	1432	13	.1933	.2828	3.328
19	1784	16	.1489	.1941	3.378
20	1760	16	.1489	.1926	3.352
21	1797	16	.1000	1.1007	3.481
22	1748	16	.0510	.0423	3.672
23	1750	16	.1028	1.1327	3.484
24	1823	16	.0511	.0425	3.671

TABLE No. 3

Summary of Data

12 inch Weir

Series 6. Height of Crest 6". Width of Channel 38"

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1746	16	.0509	.0396	3.449
2	1766	16	.0508	.0392	3.424
3	1782	16	.0974	.1009	3.276
4	1797	16	.0970	.1003	3.269
5	1762	16	.1482	.1878	3.292
6	1802	16	.1483	.1881	3.294
7	1808	16	.1966	.2858	3.279
8	1760	16	.2002	.2906	3.244
9	1766	16	.2457	.3926	3.223
10	1780.5	16	.2482	.3987	3.226
11	718	8	.5958	1.4956	3.252
12	709	7	.6016	1.5145	3.246
13	826	8	.5459	1.3000	3.223
14	805	8	.5537	1.3340	3.238
15	933	9	.5035	1.1509	3.221
16	931.5	9	.5042	1.1530	3.221
17	1139	11	.4494	.9717	3.225
18	1114.5	10	.4515	.9784	3.225
19	1767	16	.2975	.5223	3.218
20	1768	16	.3011	.5325	3.223
21	1597	14	.3556	.6830	3.222
22	1729	15	.3475	.6592	3.219
23	1361.5	12	.3990	.8130	3.226
24	1386	13	.4033	.8284	3.235

TABLE No. 3

Summary of Data

12 inch Weir

Series 7. Height of Crest 6″. Width of Channel 52″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1329	12	.4034	.8205	3.202
2	1314	12	.4023	.8172	3.202
3	1617	14	.3498	.6641	3.210
4	1576	14	.3524	.6710	3.209
5	720	7	.5989	1.4914	3.217
6	706	7	.6004	1.4976	3.219
7	814	8	.0479	.0371	3.537
8	809.5	8	.0476	.0362	3.481
9	1774	16	.1032	.1102	3.320
10	1775	16	.1028	.1093	3.322
11	1795	16	.1514	.1939	3.291
12	1800	16	.1499	.1896	3.270
13	1786	16	.2010	.2915	3.235
14	1787	16	.2004	.2877	3.207
15	1697.5	15	.2530	.4076	3.202
16	1781.5	16	.2532	.4072	3.196
17	1802	16	.2962	.5162	3.203
18	1783	16	.2956	.5139	3.198
19	1759.5	16	.5024	1.1386	3.197
20	1784	16	.5019	1.1374	3.198
21	943	9	.4502	.9689	3.207
22	944	9	.4501	.9716	3.215
23	1125.5	10	....	....	....
24	1090.5	10	....	....	....

TABLE No. 3
Summary of Data
12 inch Weir
Series 8. Height of Crest 12″ Width of Channel 12″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1812	16	.1491	.1959	3.403
2	1826	16	.0998	.1068	3.377
3	1783	16	.1488	.1945	3.389
4	1843	16	.0995	.1059	3.373
5	690	7	.5517	1.5562	3.798
6	679	7	.5541	1.5695	3.805
7	593	6	.6051	1.8109	3.847
8	573	6	.6039	1.8170	3.872
9	826	8	.4966	1.3000	3.714
10	811	8	.4957	1.2936	3.707
11	1806	16	.0471	.0365	3.572
12	1772	16	.0462	.0353	3.555
13	968	9	.4524	1.1109	3.651
14	951	9	.4524	1.1120	3.654
15	1203	11	.3964	.8926	3.576
16	1191	11	.3947	.8879	3.587
17	1503	14	.3475	.7364	3.596
18	1482	13	.3473	.7246	3.540
19	1777	16	.2972	.5672	3.501
20	1788	16	.2949	.5591	3.492
21	1773	16	.2546	.4470	3.479
22	1779	16	.2542	.4457	3.479
23	1743	16	.1961	.2983	3.435
24	1778	16	.1964	.2976	3.419

TABLE No 3
Summary of Data
12 inch Weir
Series 9. Height of Crest 12″. Width of Channel 14″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	741	7	.5499	1.4400	3.553
2	732	7	.6027	1.6940	3.620
3	634	6	.4972	1.2330	3.517
4	627	6	.4486	1.0435	3.473
5	871	8	.3953	.8505	3.423
6	828	8	.5484	1.4440	3.556
7	1029	10	.6077	1.7126	3.615
8	990	9	.4985	1.2370	3.514
9	1222	11	.4487	1.0430	3.470
10	1253	11	.3971	,8570	3.426
11	1799	16	.0480	.0385	3.660
12	1804	16	.0971	.1035	3.420
13	1754	16	.1476	.1921	3.387
14	1783	16	.1987	.2979	3.363
15	1759	16	.2507	.4212	3.356
16	1813	16	.0480	.0384	3.647
17	1745	16	.0966	.1019	3.395
18	1709	16	.1475	.1896	3.347
19	1745	16	.1980	.2950	3.349
20	1785	16	.2503	.4210	3.362
21	1601	14	.3483	.7016	3.414
22	1518	14	.2995	.5528	3.373
23	1793	16	.3506	.7081	3.411
24	1782	16	.3003	.5565	3.381

TABLE No. 3
Summary of Data
12 inch Weir
Series 10. Height of Crest 12″. Width of Channel 16″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	657	7	.6021	1.6343	3.497
2	629	6	.5526	1.4300	3.481
3	751	7	.4975	1.2025	3.427
4	735	7	.4526	1.0325	3.398
5	893	8	.6067	1.6549	3.502
6	887	8	.5546	1.4386	3.483
7	1038	10	.4954	1.1922	3.419
8	1035	10	.4536	1.0375	3.396
9	1280	12	.3986	.8591	3.386
10	1213.5	11	.3481	.6883	3.351
11	1608	14	.2984	.5463	3.351
12	1537.5	14	.2490	.4137	3.331
13	1787	16	.2028	.3045	3.333
14	1807	16	.3994	.8508	3.371
15	1796	16	.3479	.6878	3.352
16	1793	16	.2985	.5439	3.335
17	1853	16	.2498	.4147	3.323
18	1812	16	.2058	.3102	3.322
19	1784	16	.0526	.0425	3.524
20	1778	16	.0992	.1053	3.369
21	1785	16	.1455	.1854	3.340
22	1777	16	.0522	.0417	3.496
23	1779	16	.0988	.1043	3.357
24	1775	16	.1457	.1842	3.312

TABLE No. 3

Summary of Data

12 inch Weir

Series 11. Height of Crest 12". Width of Channel 20"

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1854	16	.0462	.0355	3.575
2	1824	16	.0978	.1036	3.350
3	1797	16	.1535	.1991	3.311
4	1785	16	.0459	.0352	3.580
5	1787	16	.0971	.1017	3.361
6	1808	16	.1552	.2010	3.288
7	1608	14	.3491	.6782	3.289
8	1574.5	14	.3576	.7030	3.288
9	1786	16	.3020	.5456	3.287
10	1792	16	.2497	.4103	3.288
11	1810	16	.1971	.2879	3.290
12	1804	16	.3012	.5349	3.236
13	1809	16	.2506	.4124	3.287
14	1806	16	.1974	.2886	3.291
15	683	7	.6103	1.5720	3.298
16	685	7	.5500	1.3652	3.347
17	786.5	8	.4982	1.1700	3.327
18	790	8	.4449	.9780	3.285
19	946	9	.4043	.8463	3.292
20	921.5	9	.6011	1.5670	3.363
21	1115	10	.5487	1.3590	3.344
22	1096	10	.4973	1.1653	3.323
23	1292	12	.4463	.9798	3.287
24	1301	12	.4028	.8404	3.288

TABLE No. 3
Summary of Data
12 inch Weir
Series 12. Height of Crest 12". Width of Channel 26"

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1793	16	.0979	.1011	3.301
2	1827	16	.0549	.0424	2.442
3	1793	16	.0972	.0993	3.230
4	1794	16	.0550	.0424	3.287
5	700	7	.6019	1.5340	3.285
6	705	7	.5512	1.3351	3.263
7	804	8	.4959	1.1373	3.257
8	806	8	.3581	.6915	3.227
9	945	9	.1452	.1791	3.238
10	946	9	.5991	1.5230	3.284
11	1553	14	.5503	1.3320	3.263
12	1652	15	.4959	1.1350	3.250
13	1800	16	.3580	.6898	3.220
14	1776	16	.1460	.1820	3.262
15	1091	10	.4511	.9789	3.231
16	1090	10	.3993	.8129	3.222
17	1321	12	.1985	.2867	3.242
18	1318	12	.2555	.4171	3.231
19	1784	16	.3046	.5428	3.228
20	1831	16	.4531	.9854	3.230
21	1802	16	.4002	.8148	3.218
22	1792	16	.1955	.2799	3.238
23	1765	16	.2542	.4146	3.234
24	1786	16	.2991	.5272	3.222

TABLE NO. 3

Summary of Data

12 inch Weir

Series 13. Height of Crest 12″. Width of Channel 38″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1770	16	.2532	.4104	3.222
2	1781.5	16	.2033	.2966	3.236
3	1782	16	.1525	.1941	3.259
4	1800	16	.3080	.1007	3.272
5	1814	16	.0497	.0364	3.285
6	1726	16	.2497	.4022	3.223
7	1793	16	.2029	.2953	3.231
8	1787	16	.1525	.1943	3.263
9	1808	16	.0979	.1008	3.291
19	1829	16	.0493	.0361	3.297
11	682	7	.6170	1.5620	3.223
12	701	7	.6105	1.5320	3.255
13	813	8	.5533	1.3190	3.294
14	814	8	.5536	1.3285	3.225
15	968	9	.4938	1.1093	3.157
16	658	9	.4981	1.1210	3.189
17	1034	10	.4506	.9658	3.199
18	1305	12	.4041	.8228	3.203
19	1312	12	.4021	.8125	3.210
20	1631	15	.3479	.6584	3.309
21	1610	15	.3439	.6464	3.205
22	1472	13	.2998	.5293	3.222
23	1791	16	.2994	.5257	3.209

TABLE No. 3
Summary of Data
12 inch Weir
Series 14. Height of Crest 12″. Width of Channel 52″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1800	16	.0531	.0432	3.532
2	1800	16	.0530	.0427	3.499
3	1800	16	.1000	.1063	3.362
4	1800	16	.9980	.1056	3.346
5	1800	16	.1524	.1973	3.316
6	1800	16	.1511	.1930	3.285
7	1800	16	.2048	.3021	3.259
8	1680	15	.2045	.3010	3.255
9	1800	16	.2449	.3935	3.247
10	1797.75	16	.2488	.4037	3.253
11	720	7	.5990	1 4813	3.195
12	720	7	.6000	1.4916	3.209
13	840	8	.5464	1.2954	3.208
14	840	8	.5488	1.2952	3.186
15	960	9	.5072	1.1593	3.210
16	960	9	.5006	1.1392	3.216
17	1080	10	.4447	.9489	3.260
18	1200	11	.4506	.9681	3.200
19	1333	12	.3979	.8057	3.210
20	1336.5	12	.3938	.7913	3.202
21	1547	14	.3600	.6942	3.214
22	1523	14	.3641	.7052	3.210
23	1767	16	.3077	.5516	3.232
24	1807.5	16	.2941	.5118	3.209

TABLE NO. 3

Summary of Data

12 inch Weir

Series 15. Height of Crest 18″. Width of Channel 12″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1200	9	.0579	.0489	3.510
2	1200	9	.0581	.0491	3.507
3	1200	9	.0944	.0999	3.445
4	1200	9	.0953	.1024	3.481
5	1200	9	.1483	.1947	3.409
6	1200	9	.1472	.1940	3.435
7	1200	9	.2032	.3130	3.417
8	1200	9	.2027	.3116	3.415
9	1200	9	.2503	.4271	3.411
19	1200	9	.2490	.4252	3.424
11	1200	9	.2974	.5609	3.458
12	1200	9	.2914	.5372	3.415
13	1200	9	.3544	.7278	3.450
14	1200	9	.3540	.7260	3.447
15	1200	9	.4047	.8988	3.491
16	1200	9	.4049	.8970	3.483
17	900	7	.4485	1.0558	3.514
18	900	7	.4576	1.0640	3.503
19	900	7	.4972	1.2376	3.530
20	900	7	.4927	1.2171	3.520
21	720	6	.5932	1.6364	3.582
22	720	6	.5875	1.6120	3.580

TABLE No. 3
Summary of Data
12 inch Weir
Series 16. Height of Crest 18". Width of Channel 14"

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1200	9	.0553	.0452	3.477
2	1200	9	.0557	.0465	3.536
3	1200	9	.0992	.1060	3.393
4	1200	9	.1008	.1094	3.419
5	1200	9	.1473	.1903	3.366
6	1200	9	.1449	.1843	3.341
7	1200	9	.2019	.3035	3.346
8	1200	9	.2011	.3008	3.336
9	1200	9	.2484	.4126	3.333
10	1200	9	.2484	.4122	3.330
11	1200	9	.3104	.5813	3.362
12	1200	9	.3101	.5792	3.354
13	1200	9	.3503	.6978	3.366
14	1200	9	.3469	.6840	3.348
15	1200	9	.4001	.8559	3.382
16	1200	9	.3990	.8530	3.385
17	900	9	.4581	1.0562	3.407
18	900	7	.4590	1.0594	3.406
19	900	7	.5031	1.2293	3.446
20	900	7	.5049	1.2241	3.413
21	720	6	.5984	1.5862	3.427
22	720	6	.6016	1.6030	3.436

TABLE No. 3
Summary of Data
12 inch Weir
Series 17. Height of Crest 18″. Width of Channel 16″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1200	9	.0511	.0396	3.429
2	1200	9	.0504	.0397	3.510
3	1200	9	.1016	.1090	3.365
4	1200	9	.1007	.1059	3.341
5	1200	9	.1505	.1930	3.305
6	1200	9	.1508	.1925	3.288
7	1200	9	.2004	.2944	3.281
8	1200	9	.1991	.2903	3.268
9	1200	9	.2526	.4158	3.276
10	1200	9	.2520	.4123	3.259
11	1200	9	.2982	.5328	3.273
12	1200	9	.3038	.5518	3.296
13	1200	9	.3557	0.6962	3.283
14	1200	9	.3582	.7021	3.275
15	1200	9	.4011	.8341	3.284
16	1200	9	.3997	.8253	3.266
17	900	7	.4519	1.0098	3.324
18	900	7	.4489	.9829	3.301
19	900	7	.4970	1.1578	3.304
20	900	7	.4940	1.1490	3.309
21	720	6	.6088	1.5770	3.320
22	720	6	.6052	1.5630	3.321

Table No. 3
Summary of Data
12 inch Weir
Series 18. Height of Crest 18″ Width of Channel 20″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1200	9	.0546	.0432	3.386
2	1200	9	.0554	.0451	3.458
3	1200	9	.0951	70979	3.338
4	1200	9	.0952	.0981	3.440
5	1200	9	.1528	.1965	3.290
6	1200	9	.1517	.1935	3.275
7	1200	9	.1971	.2845	3.252
8	1200	9	.2027	.2962	3.246
9	1200	9	.2456	.3957	3.252
19	1200	9	.2488	.4013	3.234
11	1200	9	.2953	.5188	3.233
12	1200	9	.2897	.4978	3.193
13	1200	9	.3421	.6467	3.232
14	1200	9	.3441	.6489	3.216
15	1200	9	.4011	.8162	3.214
16	1200	9	.4018	.8140	3.196
17	900	7	.4523	.9807	3.224
18	900	7	.4497	.9682	3.210
19	900	7	.5021	1.1426	3.212
20	900	7	.5106	1.1710	3.209
21	720	6	.5875	1.4511	3.224
22	720	6	.5747	1.4042	3.222

TABLE No. 3

Summary of Data

12 inch Weir

Series 19. Height of Crest 18″. Width of Channel 26″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1200	9	.0476	.0342	3.295
2	1200	9	.0482	.0355	3.355
3	1200	9	.0981	.1018	3.314
4	1200	9	.0968	.1002	3.327
5	1200	9	.1508	.1926	3.289
6	1200	9	.1433	.1811	3.338
7	1200	9	.2040	.2968	3.222
8	1200	9	.1936	.2773	3.256
9	1200	9	.2605	.4248	3.194
10	1200	9	.2589	.4205	3.193
11	1200	9	.2915	.4994	3.173
12	1200	9	.2927	.5046	3.186
13	1200	9	.3440	.6427	3.185
14	1200	9	.3403	.6244	3.146
15	1200	9	.4019	.8039	3.155
16	1200	9	.4009	.8038	3.167
17	900	7	.4451	.9382	3.160
18	900	7	.4493	.9480	3.148
19	900	7	.4867	1.0678	3.145
20	900	7	.5048	1.1304	3.151
21	720	7	.5901	1.4270	3.148
22	750	7	.5961	1.4525	3.157

TABLE No. 3
Summary of Data
12 inch Weir
Series 20. Height of Crest 18″. Width of Channel 38″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1200	9	.0484	.0369	3.465
2	1200	9	.0487	.0369	3.433
3	1200	9	.0975	.1013	3.328
4	1200	9	.0980	.1017	3.314
5	1200	9	.1502	.1908	3.277
6	1200	9	.1485	.1870	3.268
7	1200	9	.1931	.2766	3.260
8	1200	9	.1928	.2748	3.246
9	1200	9	.2472	.3951	3.215
10	1200	9	.2481	.3968	3.210
11	1200	9	.2961	.5160	3.203
12	1200	9	.2952	.5110	3.186
13	1200	9	.3517	.6599	3.164
14	1200	9	.3573	.6764	3.167
15	1200	9	.4097	.8296	3.164
16	1200	9	.4097	.8268	3.153
17	900	7	.4545	.9664	3.153
18	900	7	.4513	.9551	3.150
19	900	7	.4867	1.0642	3.135
20	930	7	.5109	1.1490	3.146
21	720	7	.6010	1.4579	3.130
22	720	7	.6023	1.4636	3.131

TABLE No. 3

Summary of Data

12 inch Weir

Series 21. Height of Crest 18″. Width of Channel 52″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1200	9	.0533	.0419	3.406
2	1200	9	.0538	.0433	3.469
3	1200	9	.0998	.1063	3.370
4	1200	9	.1000	.1063	3.362
5	2400	17	.1501	.1925	3.310
6	1200	9	.1494	.1909	3.306
7	1200	9	.1981	.2881	3.267
8	1200	9	.2015	.2957	3.269
9	1200	9	.2516	.4060	3.217
10	1200	9	.2544	.4122	3 213
11	1200	9	.3058	.5113	3.024
12	1200	9	.3468	.6484	3.176
13	1200	9	.3343	.6725	3.479
14	1200	9	.3968	.7891	3.158
15	900	7	.3968	.7888	3.157
16	900	7	.4459	.9384	3.152
17	900	7	.4442	.9310	3.145
18	900	7	.4988	1.1040	3.134
19	900	7	.5036	1.1180	3.128
20	720	7	.6012	1.4505	3.111
21	720	7	.5865	1.3990	3.115

TABLE No. 3
Summary of Data
12 inch Weir
Series 22. Height of Crest $26\frac{1}{2}''$. Width of Channel $12''$

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1200	9	.0565	.0474	3.529
2	1200	9	.0569	.0483	3.559
3	1200	9	.0936	.0994	3.471
4	1200	9	.0941	.1006	3.485
5	1200	9	.1492	.1973	3.423
	1200	9	.1529	.2056	3.439
7	900	7	.2069	.3208	3.408
8	900	7	.2063	.3208	3.424
9	900	7	.2469	.4196	3.420
10	900	7	.2462	.4171	3.413
11	900	7	.3013	.5660	3.422
12	900	7	.3018	.5665	3.417
13	720	7	.3500	.7150	3.452
14	720	7	.3501	.7107	3.430
15	900	7	.4040	.8842	3.443
16	960	7	.4003	.8701	3.435
17	060	7	.4467	1.0315	3.456
18	900	7	.4520	1.0480	3.448
19	900	7	.5030	1.2540	3.463
20	840	7	.5012	1.2320	3.472
21	720	7	.5943	1.5970	3.486
22	720	7	.5935	1.5930	3.485

TABLE No. 3
Summary of Data
12 inch Weir
Series 23. Height of Crest 26½″. Width of Channel 14″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1200	9	.0495	.0399	3.624
2	1200	9	.0504	.0408	3.607
3	1200	9	.1002	.1089	3.430
4	1200	9	.0983	.1058	3.433
5	1200	9	.1489	.1923	3.346
6	1200	9	.1482	.1920	3.366
7	900	7	.1972	.2931	3.347
8	900	7	.1997	.2991	3.351
9	900	7	.2534	.4251	3.334
10	900	7	.2542	.4279	3.338
11	900	7	.3056	.5689	3.368
12	900	7	.3141	.5867	3.333
13	720	7	.3464	.6862	3.365
14	720	7	.3473	.6846	3.344
15	900	7	.3965	.8371	3.352
16	900	7	.3970	.8353	3.340
17	720	7	.4487	.9998	3.327
18	720	7	.4497	1.0030	3.326
19	660	7	.4991	1.1780	3.341
20	720	7	.4879	1.1230	3.395
21	720	7	.5940	1.5370	3.357
22	720	7	.5917	1.5320	3.366

TABLE No. 3

Summary of Data

12 inch Weir

Series 24. Height of Crest 26[illegible]. Width of Channel 16

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Seconds Feet	Coefficient
1	900	7	.0517	.0454	3.864
2	900	7	.0514	.0454	3.897
3	900	7	.1039	.1188	3.620
4	900	7	.1029	.1168	3.540
5	2160	15	.1494	.1989	4.205
6	900	7	.2035	.3080	3.355
7	900	7	.2067	.3172	3.375
8	900	7	.2540	.4288	3.35[illegible]
9	900	7	.2578	.4222	3.340
10	900	7	.2947	.5314	3.321
11	900	7	.2913	.5218	3.319
12	930	7	.3441	.6684	3.311
13	720	7	.3458	.6736	3.313
14	720	7	.4092	.8645	3.303
15	720	7	.4094	.8659	3.306
16	720	7	.4564	1.0210	3.311
17	720	7	.4588	1.0290	3.311
18	720	7	.5067	1.1940	3.311
19	720	7	.5060	1.1900	3.307
20	720	7	.5946	1.5153	3.305
21	720	7	.5934	1.5128	3.310

Table No. 3
Summary of Data
12 inch Weir
Series 25. Height of Crest 26½″. Width of Channel 20′

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	900	7	.0479	.0378	3.607
2	900	7	.0489	.0392	3.626
3	900	7	.1044	.1140	3.380
4	900	7	.1024	.1109	3.384
5	720	7	.1507	.1938	3.313
6	720	7	.1500	.1914	3.294
7	720	7	.2001	.2918	3.261
8	720	7	.2023	.2947	3.239
9	720	7	.2469	.3972	3.237
10	720	7	.2460	.3993	3.273
11	720	7	.3058	.5470	3.235
12	720	7	.2970	.5207	3.912
13	720	7	.3395	.6374	3.222
14	720	7	.3401	.6364	3.209
15	720	7	.3952	.7972	3.208
16	660	7	.4004	.8028	3.168
17	720	7	.4467	.9595	3.213
18	720	7	.4481	.9587	3.195
19	720	7	.5006	1.1354	3.203
20	720	7	.4978	1.1233	3.197
21	720	7	.5807	1.4185	3.206
22	720	7	.5891	1.4422	3.190

TABLE NO. 3

Summary of Data

12 inch Weir

Series 26. Height of Crest 26½". Width of Channel 26"

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	900	7	.0537	.0426	4.228
2	900	7	.0534	.0432	3.501
3	900	7	.0942	.0980	3.390
4	900	7	.0934	.0957	3.272
5	900	7	.1567	.2039	3.287
6	900	7	.1585	.2088	3.309
7	900	7	.1945	.2788	3.250
8	900	7	.1947	.2910	3.387
9	900	7	.2449	.3926	3.239
10	900	7	.2412	.3828	3.230
11	720	7	.3031	.5377	3.221
12	720	7	.3069	.5460	3.212
13	750	7	.3446	.6466	3.196
14	750	7	.3441	.6435	3.189
15	720	7	.3997	.8040	3.182
16	720	7	.3990	.7975	3.165
17	720	7	.4507	.9592	3.170
18	720	7	.4493	.9568	3.178
19	720	7	.4978	1.1118	3.166
20	720	7	.4966	1.1110	3.174
21	750	7	.5879	1.4210	3.152
22	720	7	.5905	1.4282	3.147

TABLE No. 3
Summary of Data
12 inch Weir
Series 27. Height of Crest 26½″. Width of Channel 38″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	900	7	.0435	.0331	3.649
2	900	7	.0435	.0323	3.561
3	900	7	.0974	.1032	3.395
4	900	7	.1030	.1109	3.355
5	900	7	.1500	.1894	3.259
6	900	7	.1505	.1913	3.275
7	900	7	.1896	.2686	3.252
8	900	7	.1876	.2632	3.238
9	720	7	.2556	.4187	3.241
10	720	7	.2558	.4171	3.223
11	720	7	.2952	.5154	3.213
12	720	7	.2891	.4983	3.205
13	720	7	.3461	.6500	3.193
14	720	7	.3466	.6503	3.188
15	720	7	.4006	.8020	3.162
16	720	7	.4011	.7998	3.149
17	720	7	.4550	.9682	3.155
18	720	7	.4534	.9635	3.156
19	720	7	.4988	1.1084	3.145
20	720	7	.5005	1.1092	3.133
21	900	7	,5292	1.4127	3.669
22	720	7	.6019	1.4493	3.104

TABLE No. 3
Summary of Data
12 inch Weir
Series 28. Height of Crest 26½". Width of Channel 52

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1800	7	.0527	.0421	[illegible]
2	1800	7	.0516	.0409	[illegible]
3	1800	7	.1018	.1081	3.328
4	1800	7	.1014	.1055	3 291
5	1800	7	.1508	.1908	3.258
6	1800	7	.1489	.1877	3.267
7	1800	7	.2118	.3124	3.205
8	1800	7	.2084	.2974	3.128
9	1800	7	.2488	.3814	3.073
10	1800	7	.2956	.5147	3.202
11	1800	7	.2874	.4884	3.169
12	1800	7	.3515	.6552	3.144
13	1800	7	.3490	.6538	3.171
14	600	3	.3981	.7952	3.166
15	1140	5	.3999	.7767	3.071
16	1260	5	.3932	.7723	3.132
17	1200	5	.4512	.9428	3.114
18	1200	5	.4494	.9389	3.116
19	1200	5	.5014	1.1030	3.107
20	900	4	.6023	1.4490	3.100
21	[illegible]	4	.5996	1.4400	3.102
22	840	6	.6971	1.8050	3.098
23	840	6	.6848	1.7561	3.099
24	600	6	.2447	.3890	3.212
25	660	6	.4984	1.1183 .	3.178
26	1680	15	.0482	.0392	3.395
27	1800	16	.1056	.1167	3.144
28	1800	16	.1481	.1889	3.011
29	1200	11	.2018	.2978	3.195
30	1200	11	.2537	.4121	3.172
31	1200	11	.2997	.5198	3.149
32	840	8	.5604	1.3100	3.248
33	1200	11	.3491	.6519	3.172
34	720	7	.6042	1.4670	3.273
35	1200	11	.3965	.7841	3.180
36	1200	11	.4531	.9645	3.233
37	1080	10	.4979	1.1090	3.252

TABLE NO. 3
18 inch Weir
Series 1. Height of Crest 26½". Width of Channel 52"

Run. No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1200	11	.0585	.0755	3.5528
2	1200	11	.0574	.0740	3.5872
3	1203	11	.1050	.1782	3.4920
4	1200	11	.1053	.1760	3.4342
5	1200	11	.1548	.3077	3.3675
6	1200	11	.1545	.3048	3.3464
7	1200	11	.2082	.4699	3.2978
8	1200	11	.2070	.4653	3.2974
9	1500	13	.2543	.6279	3.2641
10	1800	16	.2528	.6224	3.2640
11	900	9	.3014	.8021	3.2320
12	900	9	.3166	.8041	3.2370
13	600	6	.4068	1.2450	3.1943
14	600	6	.4075	1.2485	3.2000
15	000	6	.4980	1.6792	3.1855
16	600	6	.4994	1.6805	3.1750
Series 2. Height of Crest 26½". Width of Channel 40"					
1	1200	11	.0514	.0836	4.7828
2	1200	11	.0514	.0826	4.7255
3	1200	11	.1003	.1653	3.4748
4	1200	11	.1006	.1638	3.4223
5	1200	11	.1538	.3041	3.3605
6	1200	11	.1528	.2989	3.3360
7	1320	12	.2059	.4637	3.3088
8	1200	11	.2052	.4604	3.3140
9	1200	11	.2529	.6235	3.2682
10	1200	11	.2512	6162	3.2629
11	960	9	.3013	.8062	3.2500
12	960	9	.2995	.7974	3.2450
13	600	6	.4071	1.2492	3.2066
14	660	7	.4101	1.2706	3.2260
15	480	5	.5081	1.7312	3.1865
16	480	5	.5116	1.7528	3.1940

TABLE No. 3

18 inch Weir

Series 3. Height of Crest 26½″. Width of Channel 30″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1200	11	.0570	.0753	3.6695
2	1200	11	.0569	.0731	3.5902
3	1200	11	.1038	.1738	3.4620
4	1320	13	.1046	.1745	3.4374
5	1260	12	.1534	.3048	3.3823
6	1200	11	.1529	.3003	3.3490
7	1200	11	.2039	.4607	3.3353
8	1200	11	.2042	.4578	3 3077
9	840	8	.2531	.6265	3.2775
10	840	8	.2520	.6228	3.2821
11	600	6	.2978	.7943	3.2580
12	600	6	.2976	.7903	3.2450
13	600	6	.4002	1.2205	3.2190
14	600	6	.3977	1.2105	3.2178
15	480	5	.5002	1.6980	3.1814
16	480	5	.4976	1.6854	3.2015

Series 4. Height of Crest 26½″. Width of Channel 24″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1200	11	.0525	.0659	3.6522
2	1200	11	.0526	.0654	3.5815
3	1200	11	.1037	.1782	3.5578
4	1200	11	.1037	.1722	3.4380
5	1200	11	.1493	.2931	3.3898
6	1200	11	.1501	.2921	3.3568
7	1200	11	.2026	.4552	3.3198
8	1200	11	.2050	.4641	3.3337
9	1200	11	.2557	.6395	3.2973
10	1200	11	.2555	.6396	3.3020
11	900	9	.3024	.8175	3.2860
12	900	9	.2993	.8058	3.2820
13	600	6	.4077	1.2756	3.2660
14	600	6	.4163	1.3138	3.2620
15	480	5	.4986	1.7193	3.2552
16	480	5	.5016	1.7355	3.2568

TABLE No. 3

18 inch Weir

Series 5. Height of Crest 26½". Width of Channel 20"

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1200	11	.0503	.0625	3.6935
2	1200	11	.0498	.0618	3.7074
3	1200	11	.0999	.1632	3.4456
4	1200	11	.1011	.2672	3.4709
5	1200	11	.1531	.3095	3.4448
6	1200	11	.1541	.3098	3.4080
7	1200	11	.2041	.4652	3.3705
8	1200	11	.2043	.4642	3.3520
9	1200	11	.2534	.6407	3.3487
10	1200	11	.2504	.6308	3.3562
11	600	6	.3049	.8458	3.3470
12	600	6	.3075	.8542	3.3388
13	600	6	.4066	1.2989	3.3405
14	600	6	.4039	1.2850	3.3370
15	480	5	.5061	1.8157	3.3617
16	480	5	.5087	1.8173	3.3398
Series 6. Height of Crest 26½". Width of Channel 18"					
1	1200	11	.0535	.0680	3.6630
2	1200	11	.0534	.0678	3.6632
3	1200	11	.0988	.1645	3.5318
4	1200	11	.0989	.1657	3.5520
5	1200	11	.1469	.2962	3.4975
6	1200	11	.1499	.3027	3.4730
7	1200	11	.2036	.4776	3.4658
8	1200	11	.2018	.4671	3.4368
9	1200	11	.2554	.6642	3.4320
10	1200	11	.2513	.6475	3.4260
11	900	9	.2992	.8018	3.2643
12	900	9	.3055	.8670	3.4330
13	600	6	.4050	1.3195	3.4128
14	600	6	.4046	1.3175	3.4130
15	540	6	.4958	1.7909	3.4205
16	480	5	.5195	1.9187	3.4160

TABLE No. 3

18 inch Weir

Series 7. Height of Crest 12″. With of Channel 52″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feed	Discharge in Second Feet	Coefficient
1	1200	11	.0512	.0648	3.7295
2	1200	11	.0514	.0675	3.8620
3	1200	11	.1018	.1725	3.5048
4	1200	11	.1022	.1707	3.4830
5	1200	11	.1535	.3038	3.3680
6	1200	11	.1539	.3036	3.3530
7	1200	11	.2039	.4443	3.2818
8	1200	11	.2037	.4453	3.1454
9	1200	11	.2532	.6188	3.2382
10	1200	11	.2540	.6300	3.2810
11	1200	11	.3059	.8351	3.2906
12	1200	11	.3032	.8285	3.3930
13	900	9	.4056	1.2653	8.2655
14	900	9	.4013	1.2441	3.2620
15	600	6	.5058	1.7820	3.3020
16	480	5	.5129	1.8110	3.2870
17	480	5	.5900	2.2460	3.3040
18	480	5	.5930	2.2582	3.2965

Series 8. Height of Crest 12″. Width of Channel 40″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feed	Discharge in Second Feet	Coefficient
1	1200	11	.0485	.0552	3.4455
2	1200	11	.0481	.0547	3.4578
3	1200	11	.0994	.1583	3.3680
4	1200	11	.1001	.1570	3.3068
5	1200	11	.1482	.2817	3 2920
6	1200	11	.1479	.2795	3 2759
7	1200	11	.2007	.4415	3.2767
8	1200	11	.2014	.4433	3.2674
9	1200	11	.2519	.6168	3.2540
10	1200	11	.2507	.6133	3 2540
11	1200	11	.3067	.8298	3 2572
12	1200	11	.3047	.8222	3.2590
13	600	6	.5133	1.8012	3.2655
14	600	6	.5109	1.7893	3.2652
15	480	5	.5936	2.2522	3 2825
16	480	5	.5956	2 2572	3 2730

TABLE No. 3
18 inch Weir
Series 9. Height of Crest 12″. Width of Channel 30″

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1200	11	.0542	.0668	3.5290
2	1200	11	.0543	.0656	3.4560
3	1200	11	.1048	.1712	3.3644
4	1200	11	.1045	.1686	3.3268
5	1200	11	.1507	.2912	3.3182
6	1200	11	.1495	.2861	3.2993
7	1200	11	.2029	.4473	3.2628
8	1200	11	.2035	.4509	3.2745
9	1200	11	.2947	.7961	3.3175
10	1200	11	.2918	.7722	3.2660
11	720	7	.4015	1.2504	3.2760
12	720	7	.3973	1.2194	3.2468
13	600	6	.4957	1.7285	3.3020
14	600	6	.4940	1.7365	3.3340
15	480	5	.5994	2.3393	3.3610
16	480	5	.6008	2.3589	3.3768
Series 10. Height of Crest 12″. Width of Channel 24″					
1	1200	11	.0453	.0495	3.4230
2	1200	11	.0450	.0478	3.3380
3	1200	11	.1011	.1592	3.3188
4	1200	11	.1016	.1601	3 2958
5	1260	11	.1532	.2955	3.2850
6	1200	11	.1530	.2961	3.2980
7	1200	11	.2141	.4863	3.2728
8	1200	11	.2139	.4867	3.2840
9	960	9	.2509	.6144	3.2746
10	960	9	.2499	.6122	3.2668
11	900	9	.3050	.8131	3.3005
12	900	9	.3008	.8166	3.3050
13	720	7	.4018	1 2695	3.3210
14	720	7	.4047	1 2181	3 3100
15	600	6	.5151	1 8982	3 4210
16	600	6	.5063	1 8355	3.3975
17	480	5	.5690	2.2315	3.4667
18	480	5	.5571	2 1510	3 4490

TABLE No. 3
18 inch Weir
Series 11. Height of Crest 12". Width of Channel 20"

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1200	11	.0546	.0726	3.7940
2	1200	11	.0546	.0723	3.7780
3	1200	11	.1014	.1716	3.5430
4	1320	12	.1017	.1698	3.4905
5	1200	11	.1518	.3080	3.4718
6	1200	11	.1522	;3071	3.4500
7	1200	11	.2039	.4737	3.4300
8	1200	11	.2056	.4775	3.4120
9	960	9	.2538	.6552	3.4160
10	960	9	.2544	,6548	3.4015
11	960	9	.3019	.8547	3.4360
12	960	9	.3047	.8675	3.4385
13	720	7	.4015	1.3239	3.4688
14	720	7	.4033	13.485	3.5100
15	600	6	.5071	1.9135	3.5348
15	600	6	.5056	1.8962	3.5170

Series 12. Height of Crest 12". Width of Channel 18"

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1200	11	.0488	.0613	3.7890
2	1200	11	.0484	.0607	3.8001
3	1200	11	.1001	.1692	3.5635
4	1200	11	.1007	.1708	3.5632
5	1200	11	.1539	.3160	3.4898
6	1200	11	.1530	.3146	3.5048
7	1200	11	.2057	.4822	3.4458
8	1200	11	.2044	.4813	3.4726
9	960	9	.2598	.6922	3.4860
10	960	9	.2599	.6914	3.4800
11	960	9	.3025	.8710	3.4905
12	960	9	.3025	.8685	3.4805
13	600	6	.4059	1.3583	3.5020
14	600	6	.4028	1.3440	3.5048
15	540	6	.5037	1.9302	3.6000
16	540	6	.4979	1.8621	3.5332

TABLE No. 3

Summary of Data

18 inch Weir

Series 13. Height of Crest 6" Width of Channel 52"

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1680	15	.0504	.0638	3.7590
2	1200	11	.0501	.0627	3.7238
3	1200	11	.1037	.1753	8.5000
4	1200	11	.1039	.1743	3.4185
5	1200	11	.1577	.3198	3.4042
6	1200	11	.1577	.3178	3.3830
7	1440	13	.2071	.4827	3.4140
8	1200	11	.2071	.4729	3.3448
9	1200	11	.2518	.6281	3.3140
10	1200	11	.2513	.6288	3.3283
11	900	9	.3046	.8306	3.3940
12	900	9	.3034	.8283	3.3048
13	600	6	.4020	1.2602	3.2960
14	660	6	.4012	1.2587	3.3018
15	610	6	.4986	1.7534	3.3205
16	600	6	.5007	1.7457	3.2835

TABLE No. 3

Summary of Data

18 inch Weir

Series 15. Height of Crest 6". Width of Channel 30"

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1200	11	.1026	.1731	3.5113
2	1200	11	.0543	.6857	3.6125
3	1200	11	.1035	.1737	3.4578
4	1200	11	.0538	.6857	3.6632
5	1200	11	.1481	.2935	3.4332
6	1200	11	.1483	.2929	3.4205
7	1200	11	.2035	.4687	3.4040
8	1200	11	.2049	.4725	3.3960

Series 14. Height of Crest 6". Width of Channel 40"

Run No.	Duration of Run in Seconds	No. of Gage Readings	Average Head Feet	Discharge in Second Feet	Coefficient
1	1200	11	.0510	.0625	3.6182
2	1200	11	.0513	.0632	3.6553
3	1200	11	.1029	.1786	3.5066
4	1200	11	.1031	.1732	3.4880
5	1200	11	.1527	.3053	3.4105
6	1200	11	.1521	.3017	3.3925
7	1200	11	.2031	.4613	3.3595
8	1200	11	.2026	.4592	3.3575
9	1200	11	.2528	.6351	3.3310
10	1200	11	.2503	.6275	3.3406
11	900	9	.3032	.8337	3.3288
12	900	9	.3025	.8294	3.3238
13	600	6	.2999	1.2558	3.[illegible]105
14	[illegible]	[illegible]	.4004	1.2603	3.3160
[illegible]	480	5	.5015	1.7722	3.3265
16	600	6	.5032	1.7650	3.2964

BULLETIN OF THE UNIVERSITY OF WISCONSIN

NO. 700

ENGINEERING SERIES, VOL. 8, NO 3, PP. 147-178

INVESTIGATION OF FLOW THROUGH FOUR-INCH SUBMERGED ORIFICES AND TUBES

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MADISON, WISCONSIN

1914

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INVESTIGATION OF FLOW THROUGH FOUR-INCH SUBMERGED ORIFICES AND TUBES

INTRODUCTION

The amount of existing information concerning the flow through submerged tubes is small. The results of experimental work given in this bulletin are presented as a slight addition to this knowledge, and while the investigations do not cover a wide range it is considered that they are worthy of presentation and it is hoped that they may prove of some value. The experimental work was done as thesis work by M. E. Allen and W. C. Parker in 1906, in the Hydraulic Laboratory of The University of Wisconsin.

In the present experiments, with the single exception of the four-inch square orifice, the areas and shapes of the cross section were kept constant, being circular with a diameter of four inches, and the effect of change in length and change in form of entrance is shown by the various curves.

APPARATUS

A general plan of the apparatus used in making the experiments is shown in Fig. 1. It consists of two adjoining tanks between the walls of which is placed the orifice or tube to be investigated. These tanks are supplied with devices for controlling the supply of water and for measuring the effects of the various conditions influencing the results.

Water Supply.—The water supply for conducting the experiments was obtained from the University mains, delivered to the

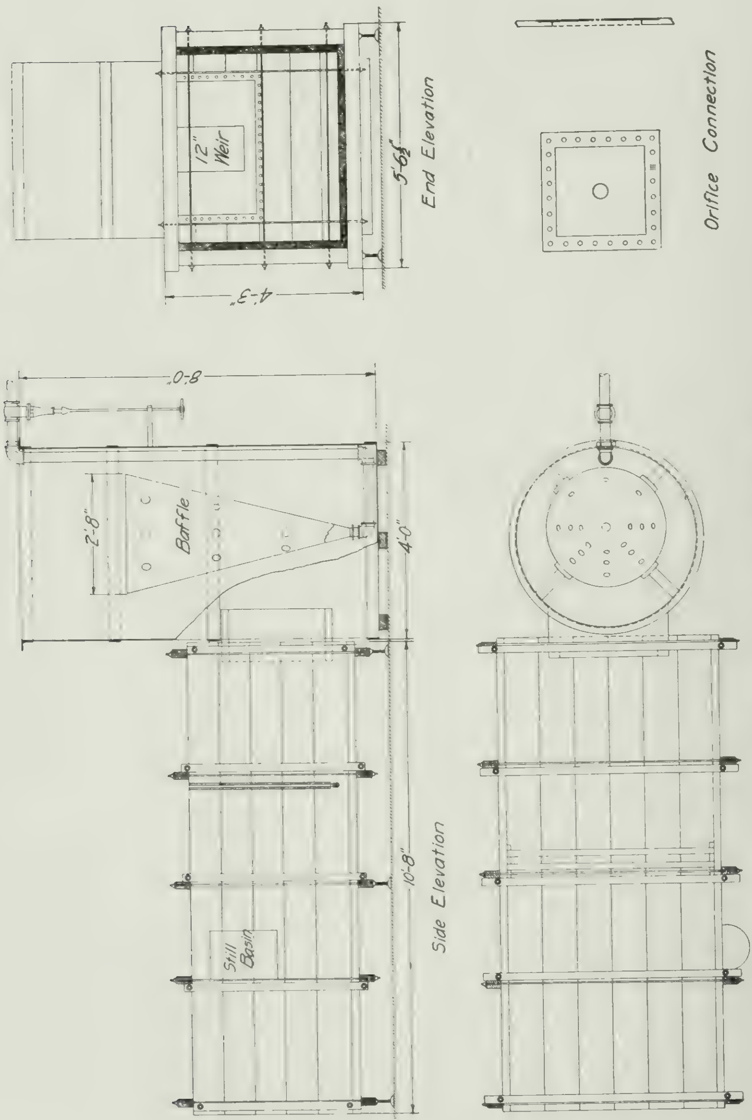

Fig. 1—General Plan of Apparatus.

head tank through a four-inch pipe. The pressure was maintained at about 82 pounds per square inch. The supply was governed and adjusted to the varying requirements by means of a valve in the four-inch pipe.

The Baffle.—This consisted of a galvanized iron cone perforated with several two-inch holes, fitted on the end of the four-inch supply pipe, and standing vertically in the head tank as shown in the drawing. This arrangement operated satisfactorily to eliminate the velocity of approach to the orifice or tube under test.

The Weir.—The weir was a sharp edged, rectangular opening in a steel plate 1/4 inch in thickness, whose crest length was twelve inches. This was set in the end of the wooden weir tank.

The Hook Gage.—The head on the weir was measured by means of an adjustable hook gage placed in a still water basin on the outside of the wooden weir basin some three feet upstream from the weir. The scale of the hook gage was fitted with a micrometer screw head and was read to one ten thousandth part of a foot. This gage is fully described by Mr. C. B. Stewart in his work on large submerged orifices and tubes published in Bulletin No. 216 of The University of Wisconsin.

The head lost in flow through the orifice is measured by means of two glass tube gages, one on the side of the head tank, and one on the side of the wooden tank. These tube gages were provided with graduated scales whose zeroes were set at the same elevation. The lost head is then taken as the difference in readings of the two gage scales. These gages were read by estimating to one thousandth part of a foot.

Orifices and Tubes.—Tests were made of the tubes and orifices given in the following tabulation:

1 circular orifice—4 inches diameter.

1 square orifice—4 inches on each side.

1 circular tube—4 inches diameter, 1 inch length, square entrance.

1 circular tube—4 inches diameter, 2 inch length, square entrance.

1 circular tube—4 inches diameter, 4 inch length, square entrance.

1 circular tube—4 inches diameter, 6 inch length, square entrance.

1 circular tube—4 inches diameter, 1 inch length, entrance bevelled 1/2"x1/2".

1 circular tube—4 inches diameter, 2 inch length, entrance bevelled, 3/4"x3/4".

1 circular tube—4 inches diameter, 4 inch length, entrance bevelled, $\frac{3}{4}''$x$\frac{3}{4}$.

1 circular tube—4 inches diameter, 6 inch length, entrance bevelled, $\frac{3}{4}''$x$\frac{3}{4}''$.

A detailed drawing of these various forms is shown in Fig. 2. In operating the apparatus, the water passed through the

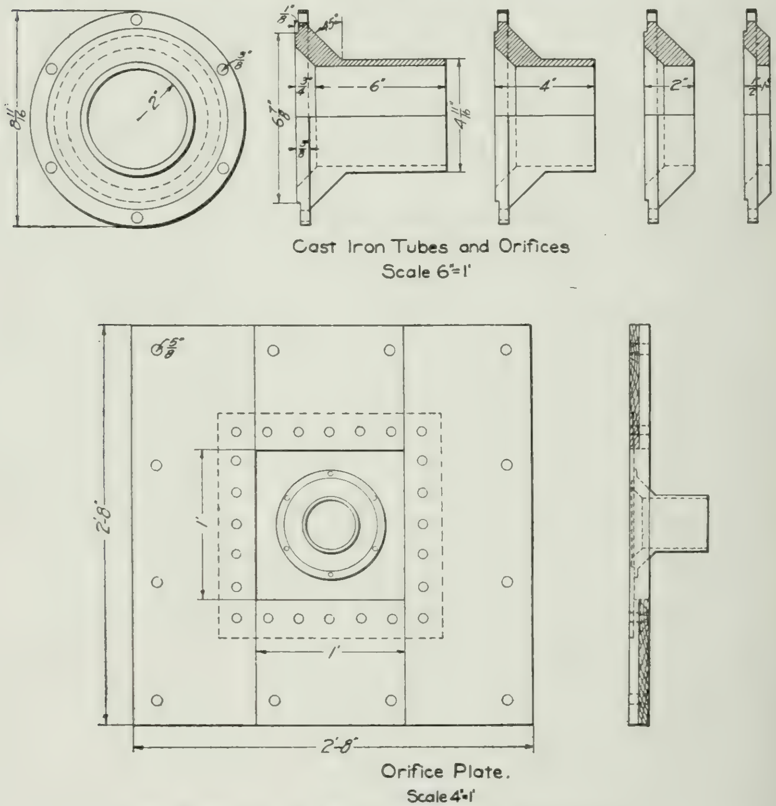

Fig. 2—Forms of Tubes used in the Experiments.

four-inch supply pipe, into and through the cone shaped baffle and thence into the head tank. From here it passed through the orifice or tube being tested into the wooden tank, the difference in elevation of the head tank and wooden discharge tank being the loss of head in flowing through the tube. The water

then flowed down the wooden tank, through a set of baffle boards and was measured by the weir fitted in the end.

Calibration of the Weir

In order to accurately measure the quantity of water flowing through the submerged tubes it was necessary to obtain the discharge curve of the weir for conditions existing during the experiments. This was accomplished by causing the discharge of the weir under known constant head to pass into a calibrated chamber for a known length of time. All gates and bulk heads connecting with the calibrated chamber were closed as tightly as possible and observations for leakage from the chamber were made at intervals varying from one hour to sixteen hours for a period of some ten days in order to determine the leakage from the chambers. The chamber has been calibrated a number of times and it is thought that the final results are accurate within one twentieth of one per cent. The data taken in the determination of leakage from the chamber is given in Table I and the results of weir calibration are given in Table II. The method of determining the volume in the calibrated chamber is as follows: The chamber is slightly greater than 160 square feet in plan so that the volume between any two readings of the gage is equal to

$$V^1 = 160\,(g_2 - g_1) + (x_2 - x_1)$$

g_2 and g_1 being the final and initial gage heights, and x_2 and x_1 the corresponding additional volume. The values of x have been carefully determined for all depths so that the actual volume in the chamber corresponding to any gage height is easily found by obtaining the value of x from the calibration curve for the chamber and adding 160 times the gage height in feet. The discharge curve of the weir, plotted logarithmically, is shown in Fig. 3.

Methods of Experiment

In testing the various orifices and tubes, each run was taken while maintaining the head as nearly constant as possible, and

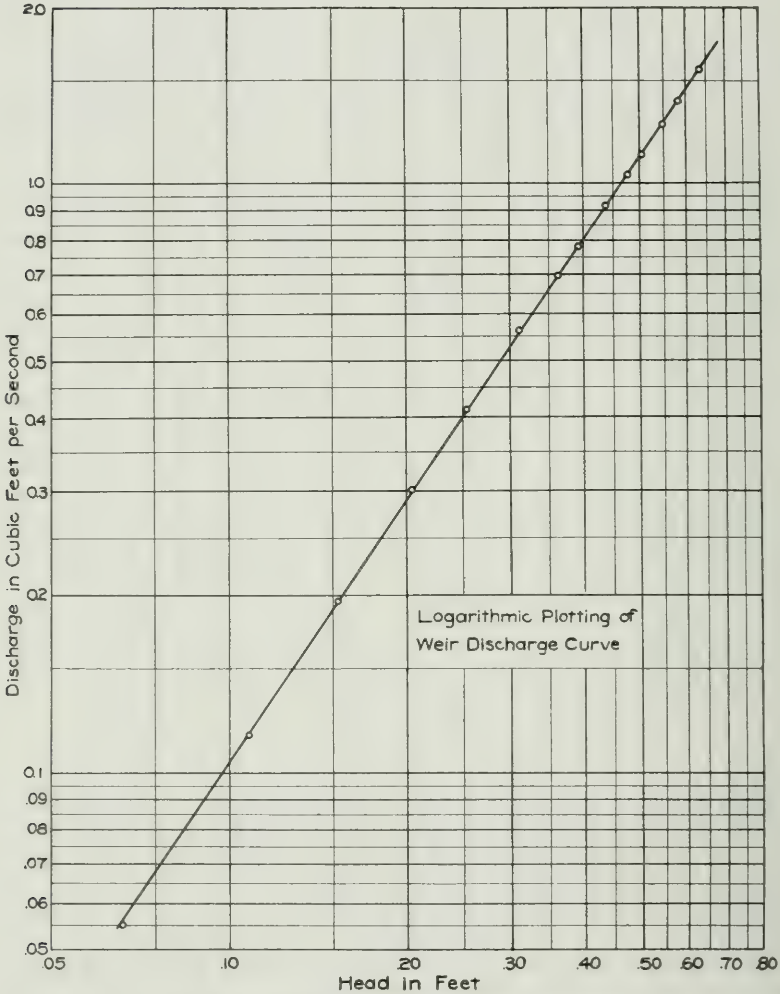

Fig. 3 -Logarithmic Plotting of Weir Discharge.

consisted of from eight to twenty readings of the various gages taken at two minute intervals. The average of all the readings constituting the run was used in computation. The averages of the runs for the various experiments are given in Tables III to XII.

Referring to the tables of data Nos. III to XII,—

Column 1 gives the number of the run

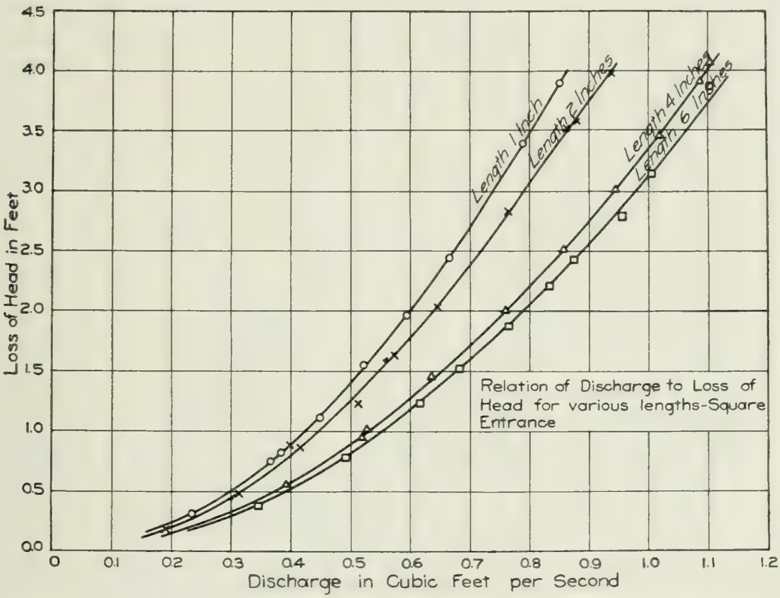

Fig. 4—Relation between Discharge and Loss of Head for the Tubes with Square Entrance.

Column 2 gives the average height of water gage in the head tank, and

Column 3 is the average height of water gage in the discharge tank. The difference is the head causing flow through the orifice or tube and is given in Column 4.

Column 5 gives the computation of theoretical discharge under the head H.

Column 6 is the average head on the weir as measured by the hook gage and the corresponding discharge, taken from the logarithmic discharge curve, Fig. 3, is given in Column 7.

Column 8 gives the values of the discharge coefficient for the various runs. It is equal to the actual discharge Q divided by the theoretical discharge, a $\sqrt{2gh}$.

The relations between the discharge in cubic feet per second and the loss of head in feet were plotted on rectangular cross section paper, Figs. 4 and 5. These curves show that, for both square and bevelled entrance conditions, the loss of head, for any one value of discharge, decreases with an increase in the length of the tube, within the limits of the experiment. The

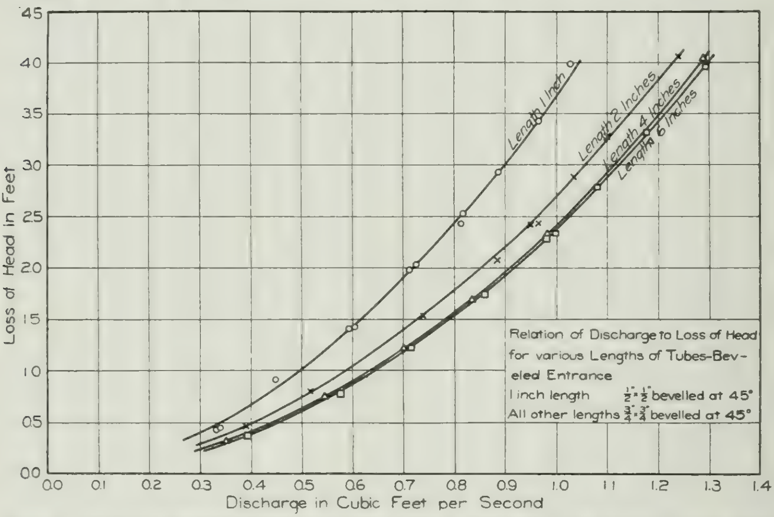

Fig. 5—Relation between Discharge and Loss of Head for the various Tubes with Bevelled Entrance.

loss of head appears to decrease with lessening rapidity as the length increases. This is shown particularly in the case of bevelled entrance where, as may be seen, there is little difference in loss for the four- and six-inch tubes.

Mr. Stewart in working on large square submerged tubes found that the loss continued to decrease with increase in tube length until the length was about 3½ times the length of one side of the square. The variation in discharge with loss of head in the orifice and tubes has been plotted on logarithmic cross-section paper in Fig. 6 and straight lines drawn averaging the plotted points.

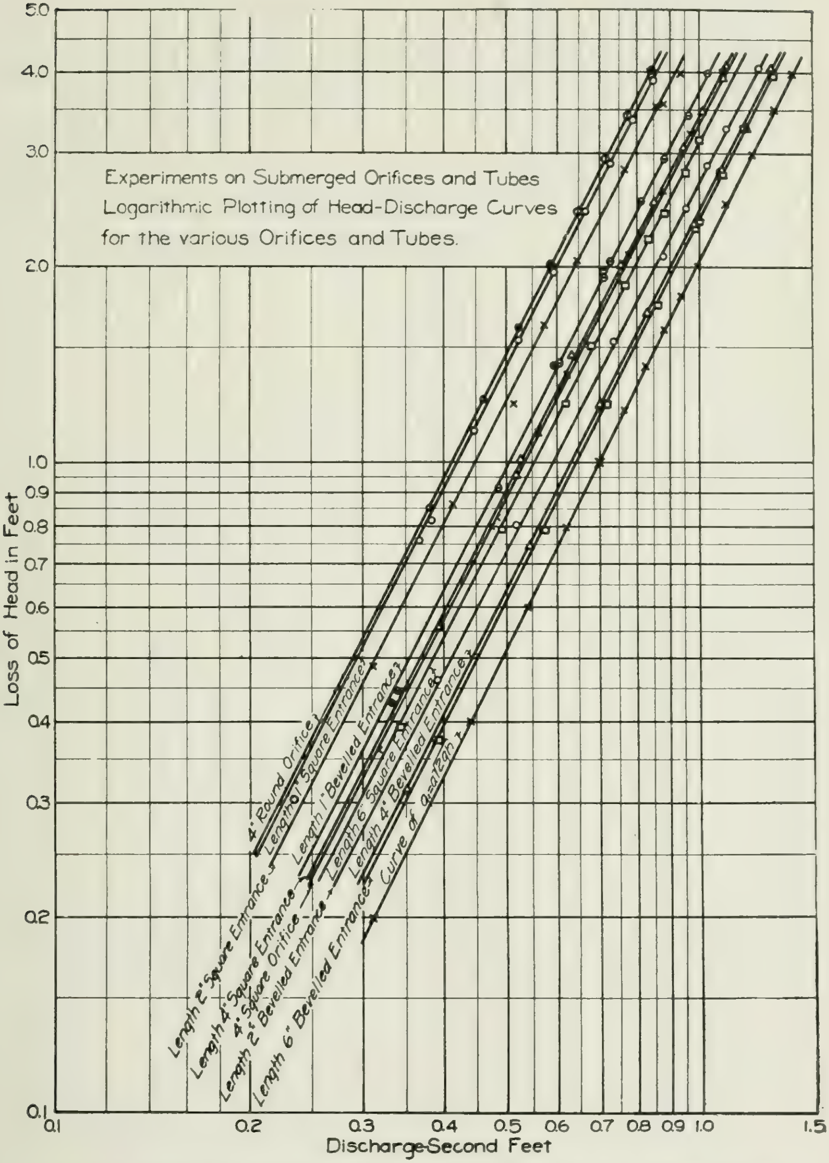

Fig. 6—Logarithmic Plotting of the Loss of Head for the Various Tubes.

Values of the discharge in cubic feet per second for even values of loss in head up to four feet were picked off and the corresponding values of the discharge coefficient were computed. The values of the coefficients so determined were plotted in their relation to loss of head in Figures 7, 8, and 9 and smooth curves drawn connecting them. The values of the coefficient of discharge as given in the data tables Nos. III to XII have been plotted on these diagrams as shown by the various symbols. The apparent discrepancies shown by the experimental values not lying on the smooth curve may be explained as follows: The

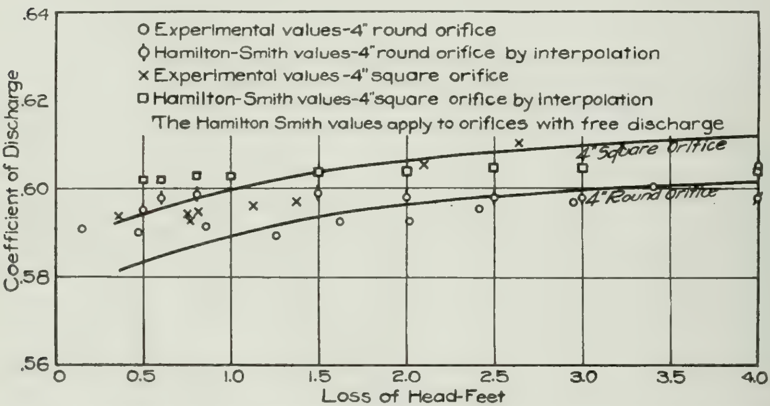

Fig. 7—Variation of the Coefficient of Discharge for the Square and Circular Orifices.

scale of the curves is necessarily large in order to show the small variation in the value of the coefficient and on the scale used values to the fourth decimal place may be plotted. In reading the value of the discharge from the logarithmic plotting the third decimal place must be estimated. These curves show that the coefficient of discharge increases with an increase in head, but that it increases with decreasing rapidity, tending to become a constant, or rather a maximum, somewhere beyond the range of the experiments. Fig. 7 shows that the coefficient is greater for a square than for a circular orifice, ranging from 0.580 to 0.616 in the former and from 0.580 to 0.601 in the latter. The coefficient derived from Hamilton Smith's tabulations applicable

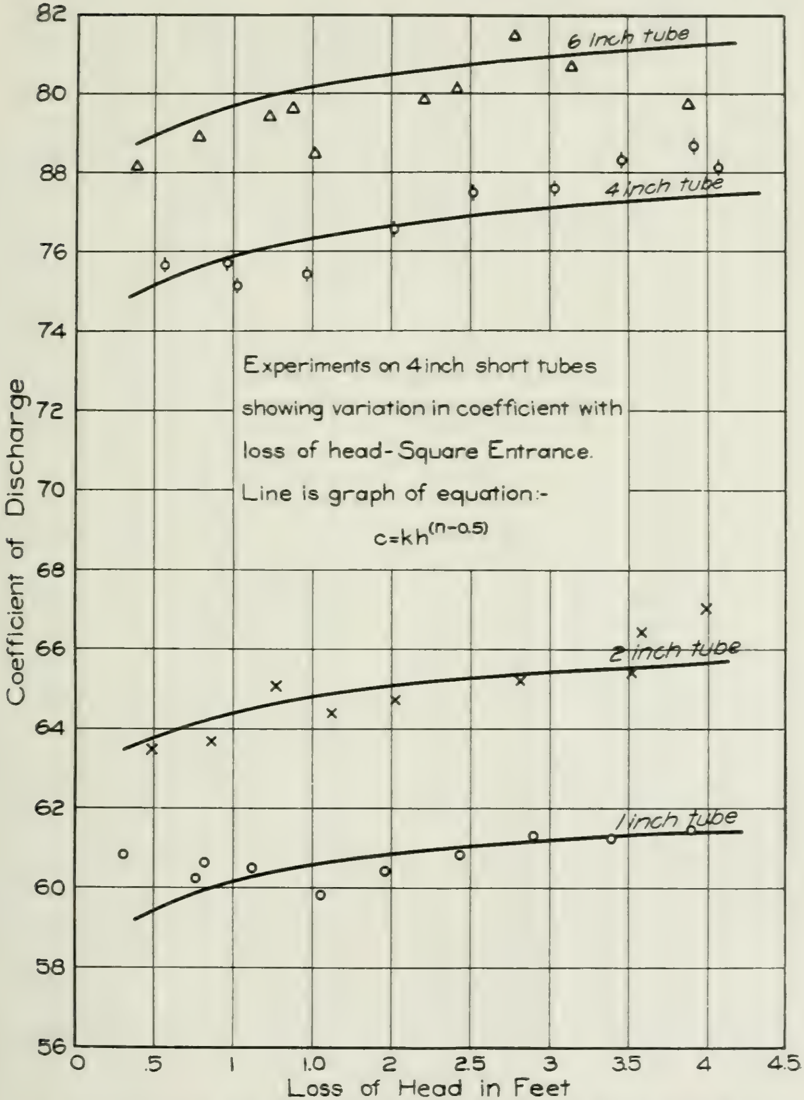

Fig. 8—Variation of the Coefficient of Discharge for the Tubes with Square Entrance.

to orifices with free discharge have been platted on this diagram. The variation is similar but the range is not so great as is found in the present experiments with submerged orifices.

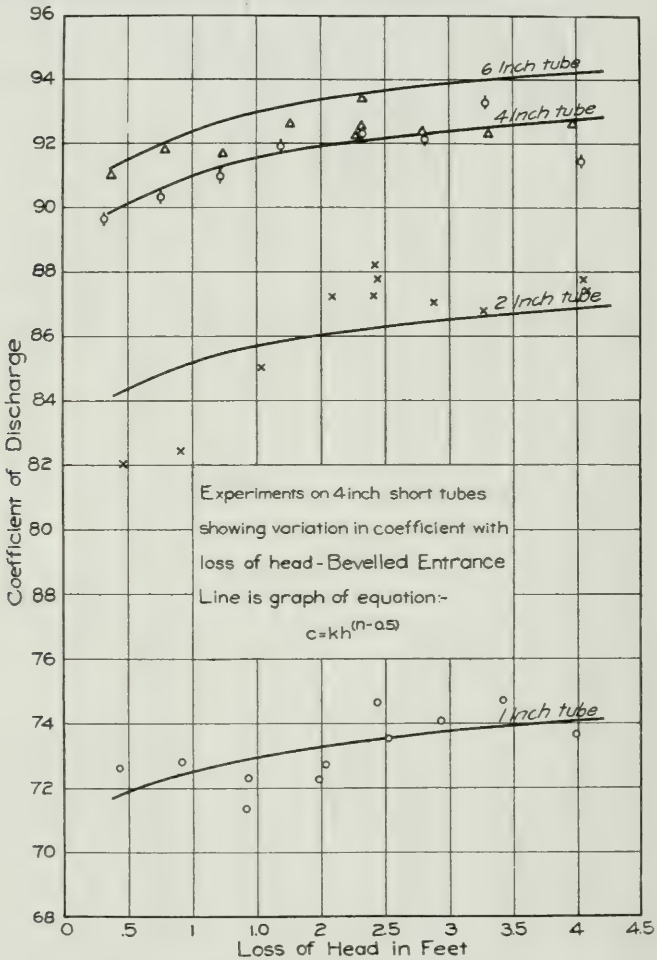

Fig. 9—Variation of the Coefficient of Discharge for the Tubes with Bevelled Entrance.

Figs. 8 and 9 show that the value of the coefficient increases with increase in head, and with increase in length of tube within the limits of the experiments. They also serve to show that

bevelling the entrance increases the value of the discharge coefficient considerably.

This is shown more clearly in Fig. 10, which gives the variation in value of the discharge coefficient, with the length of tube for the two conditions of entrance and for two values of the loss of head. The difference in loss between square and bevelled

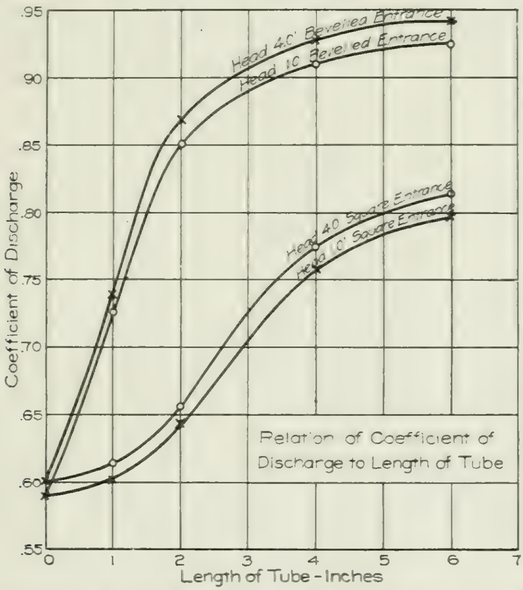

Fig. 10 Variation of the Coefficient of Discharge with Length of Tube.

entrance seems to tend to become a constant somewhat beyond the experimental limits, probably in the vicinity of length equals two diameters. The curves given in Figs. 7, 8, and 9 are not the same in shape as those given by Mr. Stewart showing similar relations on the large square tubes, the latter approaching a hyperbolic form rather than the parabola as given in the present work.

CONCLUSIONS

The conclusions which may be reached from the results of these experiments, may be summarized as follows:

(a) The coefficient of discharge increases somewhat with increase in the head causing flow through the orifice or tube.

This is contrary to our experience with orifices and tubes discharging into the air. In such experiments it is found that the value of the coefficient decreases as the head increases and tends to become a constant value for the higher heads. This point will be more fully discussed later.

(b) The coefficient of discharge increases with increase in the length until the length is at least two diameters, this increase becoming less rapid as the length of tube is further increased and it is probable that a maximum value of the coefficient would be reached when the length of tube is equal to 3.5 or 4 diameters.

This effect is to be expected since experiments with orifices and short tubes discharging into the air show that the coefficient under those conditions is similarly affected. The addition of tube lengths of 2 to 3 diameters to a sharp edged orifice results in increasing the coefficient of contraction from approximately 0.61 to 0.82. That is to say, the minimum cross section of the issuing jet is increased from 62 to 82 per cent. of the area of the orifice by the addition of the short tube lengths. When a length of tube is added, beyond that at which the maximum effect in suppressing the contraction is obtained, frictional resistance in the tube itself becomes the controlling factor contributing to the loss of energy and a reduction in the value of the coefficient of discharge results.

(c) The value of the coefficient of discharge is increased by bevelling the entrance to the tube. This is in accordance with previous observations and is to be expected, since any variation from a sharp edge acts to decrease the contraction of the issuing stream and will therefore tend to increase the discharge.

(d) The value of the coefficient of discharge is less for a circular orifice than for a square orifice whose side is equal to the

diameter of the circle. This conclusion is in accordance with results of other experiments.

A comparison of Hamilton Smith's tables of coefficients for circular and square orifices shows that the value of the coefficient of discharge for the square is always greater than that for the circular orifice. This is probably due to the effect of the edges of the orifice near the corner of the square, mutually interacting to tend to reduce the full contraction at these places.

It is stated in the discussion of the variation in values of the coefficients that they seemed to vary according to a different law than that found by Mr. Stewart. The reason for this is not apparent but is probably due to conditions peculiar to the apparatus used in the two cases, since the general law of variation was consistently the same for all tubes used throughout the investigation.

A study of the results in connection with the results of later work on the same apparatus seems to indicate that conditions due, in part at least, to the depth of submergence probably had an effect upon the value of the coefficient of discharge in this particular investigation. In the later experimental work, which was suggested by and carried out under the direction of Professor Charles I. Corp, the tank was provided with an overfall baffle wall of an adjustable height, by means of which the depth of submergence on the tube under test could be maintained constant during a range of heads. Two series of runs on the tube which is one inch in length with entrance bevelled 1/2 inch by 1/2 inch, were made, keeping the submergence constant during each run at 1.78 feet and 1.27 feet respectively. The values of the coefficient of discharge $C = \frac{Q}{a \; 2gh}$ were computed for each run and plotted against the corresponding value of the head. The two curves thus obtained are similar and more nearly of the shape of those obtained with orifices having free discharge and lie a considerable distance apart, the curve for the higher submergence giving greater values for the coefficient. These curves are shown in Fig. 11. The values for the submergence during the original experiments on this tube were obtained from measurements made later and are shown by the figures near the points in Fig. 11.

The points determining the middle curve shown were obtained by increasing the values of C as given in the original data by a proportional amount, depending upon the difference in submergence to interpolate values corresponding to a uniform submergence of 1.50 feet. In the original experiments, the depth of submergence increased with the head, and it is to be noted that those under the greatest head are the ones showing

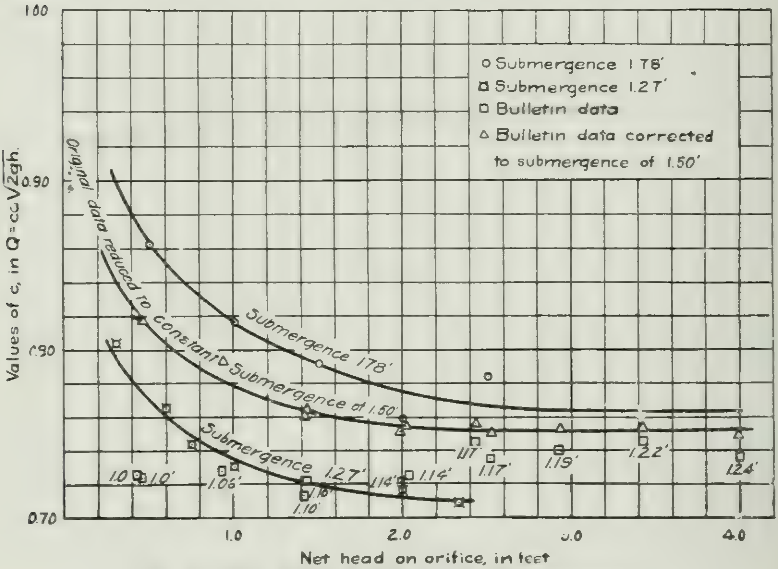

Fig. 11—The Effect of the Depth of Submergence.

the greatest variation from a uniform curve corresponding to that submergence. Such comparison as has been attempted can be nothing but tentative, particularly when based upon the meagre data at hand. It would seem to indicate, however, that irregular and disturbed conditions of flow tend to increase the value of the coefficient of discharge above its normal value, since it seems probable that the variation in C is more largely due to these causes than to the mere depth of submergence. Further, it seems that the peculiar shape of the head-coefficient curves obtained is due to the influence on the flow exerted by the particular shape and size of the channel used, rather than to any characteristic of the tube tested.

The apparent explanation of the increase in the value of the coefficient of discharge is that the currents of water caused by the shape of the channel have a tendency to reduce the head acting on the lower or outlet end of the tube, making the effective head greater than the observed differences in water levels.

In view of the later results it would appear that the value of the original data for actual comparative purposes is small, but the experience gained along this line of work is valuable since it throws light upon some of the points which may affect the reliability of laboratory data when taken by use of small and inadequate apparatus.

It is probable that the discharge from any submerged orifice or tube is influenced by the disturbance on the downstream side caused by the issuing jet. This agitation and consequently its influence would be modified by the depth of water or submergence of the outlet end of the tube, by the distance to the sides and bottom of the channel, by any obstruction in the channel itself and probably by other conditions, such as the cross sectional shape, and alignment of the channel.

The problem of measuring the downstream head under the disturbed condition of flow is also a serious one and it is doubtful whether even reasonably accurate results may be expected in using a submerged orifice or tube as a measuring device, where calibration cannot be made under conditions actually existing in the particular case.

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TABLE I

MEASUREMENT OF LEAKAGE FROM CALIBRATED CHAMBER

Time, seconds.	Depth in Chamber.			Average head, feet.	Correction.			160 D. cu. ft.	Volume leakage, 160 D + x.	Rate of leakage, cu. ft. sec.
	Initial.	Final.	D. Diff.		x_1	x_2	x cu. ft.			
50,400	2.263	2.151	0.112	2.207	11.85	11.27	0.58	17.92	18.50	.000367
25,500	4.172	4.049	0.123	4.110	21.83	21.14	0.69	19.68	20.37	.000799
49,980	3.922	3.703	0.219	3.812	20.46	19.32	1.14	35.04	36.18	.000724
57,600	5.683	5.368	0.315	5.525	29.90	28.20	1.70	50.40	52.10	.000904
8,820	5.372	5.321	0.051	5.347	28.22	27.95	0.27	8.16	8.43	.000956
4,200	6.930	6.896	0.034	6.913	36.55	36.39	0.16	5.44	5.60	.001330
57,600	6.581	6.244	0.337	6.413	34.72	32.87	1.85	53.92	55.77	.000968
54,000	7.756	7.272	0.484	7.514	40.92	38.38	2.54	77.44	79.98	.001481

TABLE II
DATA ON CALIBRATION OF WEIR

Average Head on Weir feet.	Depth in Chamber.			160 D	Correction.			160 D + x V¹ cu. ft.	Ave. Depth in Chamber Feet.	Leakage per. sec. cu. ft.	Volume of Leakage cu. ft.	Duration of Run Seconds	V¹ plus Leakage cu. ft.	Discharge of Weir cu. ft. per. sec.
	Initial.	Final.	Diff D		x_1	x_2	Diff. x cu. ft.							
.0661	1.274	3.261	1.987	317.92	7.05	17.00	8.95	327.87	2.267	.00038	2.28	6000	330.15	.055025
.1083	2.624	6.871	4.193	670.88	13.60	36.00	22.40	693.28	4.720	.000896	5.376	6000	693.66	.1156
.1533	0.373	6.773	6.400	1024.00	2.72	35.75	33.03	1057.03	3.573	678	3.68	5439	1060.71	.1950
.2048	0.745	6.700	5.955	952.80	4.52	35.30	30.78	983.58	3.722	688	2.26	3278	985.84	.3008
.2531	1.425	7.500	6.075	972.00	7.75	39.60	31.85	1003.85	4.463	843	2.06	2441	1005.91	.4121
.3111	0.260	7.500	7.24	1158.40	2.20	39.60	37.40	1195.80	3.880	724	1.54	2131	1197.34	.5619
.3627	0.600	7.600	7.00	1120.00	3.80	40.10	36.30	1156.30	4.100	768	1.274	1659	1157.57	.6977
.3932	0.600	7.600	7.00	1120.00	3.80	40.10	36.30	1156.30	4.100	768	1.14	1483	1157.44	.7804
.4385	0.700	7.700	7.00	1120.00	4.30	40.65	36.35	1156.35	4.200	789	0.996	1261	1157.35	.9178
.4768	0.600	7.600	7.00	1120.00	3.80	40.10	36.30	1156.30	4.100	768	0.86	1119	1157.16	1.0341
.5070	0.600	7.600	7.00	1120.00	3.80	40.10	36.30	1156.30	4.100	768	0.79	1027	1157.09	1.1266
.5477	1.100	7.800	6.70	1072.00	6.21	41.20	34.99	1106.99	4.450	841	0.74	875	1107.73	1.266
.5835	0.600	7.600	7.00	1120.00	3.80	40.10	36.30	1156.30	4.100	768	0.64	833	1156.94	1.389
.3223	4.600	7.400	2.80	448.00	24.38	39.05	14.67	462.67	6.00	.001170	0.85	729	463.52	0.6358
.4134	4.100	6.300	2.20	352.00	21.40	33.20	11.80	363.80	5.20	.001000	0.43	429	364.23	0.849
.4636	1.300	4.300	3.00	480.00	7.19	22.70	15.51	495.51	2.80	.000492	0.24	496.8	495.75	0.998
.5082	2.900	5.900	3.00	480.00	15.03	31.04	16.01	496.01	4.40	831	0.36	436.8	498.37	1.113
.5636	3.400	6.900	3.50	560.00	17.70	36.40	18.70	578.70	5.15	988	0.435	439.8	579.13	1.317
.5842	1.400	6.600	5.20	832.00	7.62	34.80	27.18	889.18	4.00	748	0.463	619	859.64	1.389
.6360	1.500	7.300	5.80	928.00	8.12	38.30	30.18	958.18	4.40	831	0.508	612	958.69	1.568

TABLE III

SUBMERGED ORIFICE—CIRCULAR—DIAM. 4 INS. IN ¼ IN. STEEL PLATE—SHARP EDGES

Run No.	Orifice Gages. 1	2	Diff. H Feet.	$a\sqrt{2gH}$	Weir. h	Q Sec ft.	C $\frac{Q}{a\sqrt{2gH}}$
1	2.652	2.5086	0.145	.2667	.1319	0.157	.5909
2	3.042	2.573	.469	.5902	.1972	.283	.5902
3	3.469	2.617	.851	.5912	.2410	.382	.5912
4	3.903	2.649	1.254	.5893	.2754	.462	.5893
5	4.284	2.672	1.612	.5921	.2999	.524	.5921
6	4.705	2.692	2.012	.5924	.3236	.588	.5924
7	5.136	2.715	2.421	.5954	.3455	.648	.5954
8	5.689	2.742	2.949	.5967	.3700	.716	.5967
9	6.170	2.760	3.410	.6006	.3908	.776	.6006
10	6.793	2.777	4.015	.6054	.4155	.848	.6054

TABLE IV

SUBMERGED ORIFICE—SQUARE—4 INS. × 4 INS. IN ¼ IN. STEEL PLATE SHARP EDGES

Run No.	Orifice Gages. 1	2	Diff. H. Feet.	$a\sqrt{2gH}$	Weir. h	Q Sec. ft.	C $\frac{Q}{a\sqrt{2gH}}$
49	2.957	2.594	0.363	0.537	.2135	0.319	.5940
50a	3.401	2.651	.750	.771	.2723	.458	.5940
50b	3.423	2.653	.771	.782	.2766	.464	.5932
51	3.472	2.646	.826	.809	.2833	.484	.5982
51a	3.449	2.644	.805	.800	.2805	.476	.5950
52	3.824	2.690	1.134	.948	.3145	.565	.5960
53	4.069	2.698	1.371	1.042	.3369	.622	.5970
54	4.862	2.765	2.097	1.288	.3920	.780	.6056
55	5.430	2.795	2.636	1.445	.4253	.882	.6105
56	6.033	2.813	3.220	1.598	.4565	.974	.6095
57	6.815	2.839	3.975	1.773	.4926	1.090	.6148

TABLE V

SUBMERGED TUBE—DIAM. 4 INS.—LENGTH 1 IN.—SQUARE ENTRANCE

Run No.	Tube Gages. 1	Tube Gages. 2	Tube Gages. Diff. H	$a\sqrt{2gH}$	Weir. h	Weir. Q Sec. ft.	C $= \frac{Q}{a\sqrt{2gH}}$
30	2.856	2.552	0.304	.3864	.1734	.235	.6082
31a	3.373	2.612	.761	.6111	.2351	.368	.6024
31b	3.438	2.618	.8194	.6335	.2422	.384	.6062
32	3.762	2.644	1.118	.7406	.2689	.448	.605
33	4.222	2.674	1.548	.8708	.2991	.521	.5983
34	4.665	2.700	1.965	.9814	.3252	.593	.6043
35	5.164	2.725	2.438	1.092	.3506	.664	.6082
36	5.642	2.747	2.895	1.1907	.3740	.730	.6131
37	6.156	2.766	3.390	1.289	.3945	.789	.6121
38	6.775	2.875	3.900	1.3845	.4169	.850	.6150

TABLE VI

SUBMERGED TUBE. DIAM. 4 INS. LENGTH 2 INS. SQUARE ENTRANCE

Run No.	Tube Gages. 1	Tube Gages. 2	Tube Gages. Diff. H	$a\sqrt{2gH}$	Weir. h	Weir. Q Sec. ft.	C $\frac{Q}{a\sqrt{2gH}}$
21	2.702	2.530	0.172	.2905	.1504	.191	.6576
22	3.075	2.583	0.489	.490	.2109	.311	.6348
23	3.499	2.631	0.869	.6531	.2555	.416	.6370
24	3.945	2.669	1.277	.7903	.2948	.515	.6517
25	4.317	2.691	1.625	.8925	.3188	.575	.6443
26	4.752	2.716	2.036	.9989	.3447	.645	.6457
27	5.574	2.752	2.823	1.1760	.3870	.765	.6505
28a	6.306	2.777	3.529	1.3146	.4196	.860	.6542
28b	6.361	2.783	3.577	1.3237	.4239	.880	.6648
29	6.797	2.802	3.995	1.3986	.4443	.938	.6707

TABLE VII

SUBMERGED TUBE–DIAM. 4 INS. LENGTH 4 INS. SQUARE ENTRANCE

Run No.	Tube Gages 1	Tube Gages 2	Tube Gages Diff. H	$a\sqrt{2gH}$	Weir h	Weir Q Sec. ft.	C $= \frac{Q}{a\sqrt{2gH}}$
39	3.185	2.628	0.557	.5222	.24765	0.395	.7565
40	3.635	2.675	.960	.6867	.2986	.520	.7573
41	4.165	2.715	1.451	.8421	.3405	.635	.7541
42	4.761	2.758	2.003	.9912	.3847	.759	.7657
43	5.292	2.788	2.504	1.1074	.4182	.858	.7749
44	5.862	2.818	3.043	1.2194	.4471	.946	.7759
45	6.301	2.836	3.465	1.302	.4692	1.020	.7835
46	6.774	2.855	3.917	1.3846	.4926	1.090	.7874
47	6.938	2.860	4.078	1.414	.4969	1.105	.7815
48	3.688	2.685	1.003	7.014	.3018	0.527	.7513

TABLE VIII

SUBMERGED TUBE—DIAM. 4 INS.—LENGTH 6 INS. SQUARE ENTRANCE

Run No.	Tube Gages. 1	Tube Gages. 2	Tube Gages. Diff H	$a\sqrt{2gH}$	Weir. h	Weir. Q Sec. ft.	C $= \frac{Q}{a\sqrt{2gH}}$
11	2.990	2.599	0.390	.4375	.2229	0.342	.7817
12	3.447	2.661	.786	.6209	.2853	.489	.7890
13	3.933	2.706	1.227	.7756	.3325	.516	.7942
14	4.627	2.756	1.871	.9576	.2854	.762	.7958
15	4.254	2.729	1.524	.8645	.3572	.678	.7850
16	4.981	2.775	2.205	1.0395	.4087	.830	.7984
17	5.2127	2.7939	2.418	1.0892	.4225	.873	.8015
18	5.606	2.812	2.793	1.169	.4499	.955	.8170
19	5.988	2.839	3.148	1.2404	.4656	1.001	.8070
20	6.738	2.855	3.881	1.379	.4967	1.100	.7977

TABLE IX

SUBMERGED TUBE—DIAM. 4 INS. LENGTH 1 IN. BEVELLED ENTRANCE—¼ IN. · ¼ IN.

Run No.	Tube Gages. 1	Tube Gages. 2	Tube Gages. Diff H.	$a\sqrt{2gH}$	Weir. h	Weir. Q Sec. ft.	$C = \frac{Q}{a\sqrt{2gH}}$
58a	3.023	2.595	0.428	.4585	.2201	.333	.7264
58b	3.043	2.598	.446	.4676	.2218	.339	.7250
59	3.576	2.659	.917	.6703	.2850	.488	.7282
60a	4.136	2.702	1.434	.8379	.3304	.606	.7232
60b	4.104	2.700	1.406	.8309	.3258	.593	.7137
61a	4.720	2.739	1.981	.9856	.3692	.712	.7225
61b	4.782	2.744	2.039	.9996	.3731	.727	.7273
62	5.291	2.771	2.519	1.1116	.4047	.818	.7357
63	5.725	2.794	2.931	1.1984	.4281	.888	.7411
64	6.242	2.818	3.424	1.295	.4529	.968	.7475
65	6.827	2.837	3.990	1.3986	.4755	1.030	.7365
66	5.208	2.774	2.435	1.092	.5036	.815	.7465

TABLE X

SUBMERGED TUBE—DIAM. 4 INS.—LENGTH 2 INS. ENTRANCE BEVELLED ¾ IN. × ¾ IN.

Run No.	Tube Gages. 1	Tube Gages. 2	Tube Gages. Diff. H	$a\sqrt{2gH}$	Weir. h	Weir. Q Sec. ft.	C $\frac{Q}{a\sqrt{2gH}}$
67	3.080	2.617	0.463	.4767	.2454	0.391	.8203
68	3.469	2.667	.802	.6272	.2966	.517	.8246
69	4.283	2.744	1.539	.868	.3774	.738	.8504
70	4.873	2.790	2.083	1.0108	.4259	.882	.8727
71a	5.261	2.815	2.446	1.0934	.4503	.960	.8781
71b	5.245	2.815	2.430	1.0913	.4510	.963	.8824
71c	5.234	2.814	2.420	1.0899	.4489	.950	.8725
72	5.716	2.835	2.881	1.1886	.4763	1.035	.8708
73	6.137	2.863	3.274	1.267	.4965	1.100	.8682
74a	6.934	2.871	4.063	1.4112	.5373	1.240	.8786
74b	6.953	2.871	4.082	1.4164	.5371	1.238	.8741

TABLE XI

SUBMERGED TUBE—DIAM. 4 INS. LENGTH 4 INS. ENTRANCE BEVELLED ¾ IN. × ¾ IN.

Run No.	Tube Gages. 1	Tube Gages. 2	Tube Gages. Diff. H	$a\sqrt{2gH}$	Weir. h	Weir. Q Sec. ft.	C $\frac{Q}{a\sqrt{2gH}}$
83	2.920	2.606	0.314	.3927	.2288	0.352	.8964
84	3.436	2.684	.752	.6076	.3088	.549	.9036
85	3.965	2.739	1.226	.7749	.3659	.705	.9099
86	4.474	2.783	1.691	.910	.4109	.836	.9190
87	5.149	2.826	2.324	1.0675	.4596	.985	.9227
88	5.667	2.856	2.710	1.1725	.4898	1.080	.9212
89	6.179	2.882	3.296	1.2705	.5203	1.185	.9327
90	6.955	2.910	4.045	1.407	.5519	1.290	.9170

TABLE XII

SUBMERGED TUBE—DIAM. 4 INS. LENGTH 6 INS. ENTRANCE BEVELLED ¾ IN. × ¾ IN.

Run No.	Tube Gages. 1	2	Diff. H	$a\sqrt{2gH}$	Weir. h	Q Sec. ft.	C $\frac{Q}{a\sqrt{2gH}}$
75	2.997	2.623	0.3745	.4284	.2449	0.390	.9104
76	3.481	2.692	.788	.6214	.3186	.574	.9180
77	3.972	2.7405	1.231	.77665	.3691	.712	.9168
78	4.541	2.788	1.753	.9268	.4180	.858	.9258
79a	5.122	2.824	2.298	1.0612	.4570	.978	.9215
79b	5.155	2.827	2.329	1.0685	.4630	.999	.9340
79c	5.130	2.824	2.306	1.06295	.4586	.981	.9230
80	5.654	2.853	2.799	1.1711	.4890	1.080	.9223
81	6.193	2.884	3.309	1.2733	.5191	1.175	.9231
82	6.874	2.914	3.960	1.3930	.55396	1.296	.9260

TABLE XIII

VALUES OF COEFFICIENTS AND EXPONENT IN EXPONE TIAL FORMULA FOR DISCHARGE

Tube or Orifice.	Length. inches.	Values of P and n in the exponential formula $Q = Ph^n$ — n	P	Values of K in the formula $C = Kh^{(n-0.5)}$ where $K = \frac{P}{A\sqrt{2g}}$
4″ Circular orifice............	....	0.514	0.412	0.590
4″ Square orifice..............	....	0.514	0.534	0.600
Tube—Square entrance.......	1	0.514	0.420	0.602
" " "	2	0.514	0.550	0.644
" " "	4	0.514	0.530	0.759
" " "	6	0.514	0.557	0.797
Tube—bevelled entrance.....	1	0.514	0.507	0.726
" " "	2	0.514	0.594	0.852
" " "	4	0.514	0.635	0.910
" " "	6	0.514	0.645	.0.92

BULLETIN OF THE UNIVERSITY OF WISCONSIN

NO. 810

ENGINEERING SERIES VOL. 8, NO. 4, PP. 179-246

HIGH VERSUS LOW ANTENNAE IN RADIO TELEGRAPHY AND TELEPHONY

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MADISON, WISCONSIN

September, 1916

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HIGH VERSUS LOW ANTENNAE IN RADIO TELEGRAPHY AND TELEPHONY

I. PREFATORY CONSIDERATIONS

1. Current Notions Relating to the Height of Radio Antennae

In a paper presented before the British Institution of Electrical Engineers in 1899, Marconi announced a relation between the working telegraphic distance of a pair of wireless stations and the height of the station antennae. This relation, which has come to be known as Marconi's law, is as follows: For stations with antennae of equal height, "the distance at which signals can be obtained varies approximately with the square of the distance of the capacities from earth, or perhaps with the square of the length of the vertical conductors."*

This relation, which is based upon experiments between stations with antennae each consisting of a single vertical wire or a single wire connected to a capacity area of very moderate dimensions, has exerted and now exerts a guiding or dominating influence in the practice of wireless telegraphy. The great elevation—from 100 to 1000 feet—at which the capacity areas are mounted in all wireless stations is conclusive evidence either of the necessity or of the importance of high antennae in the minds of those practicing the art of wireless telegraphy at the present time.

From the very early years of the art, the practice in the construction of wireless antennae has been to use, not single wires, but a multiplicity of wires arranged in the form of a fan, a harp, an umbrella, a cylindrical cage, or an inverted cone or pyramid. These wires constitute an extended "capacity area," and, as previously stated, the practice is

*Marconi, *Wireless Telegraphy—Jour. of the Inst. of Electrical Engineers* 1899 Vol. 28, page 279.

to mount this extended capacity area, or at least a large part of it, at a great elevation above the earth. For example, the capacity area of the government station at Arlington, D. C., consists of three approximately horizontal wire harps suspended at a mean elevation of about 500 feet above the surface of the earth. Each harp contains 23 strings and has a width of 88 feet and a mean length of 300 ft.

The following elementary considerations seemed to the writer to warrant a critical examination of the practice of mounting such extended capacity areas at these great elevations.

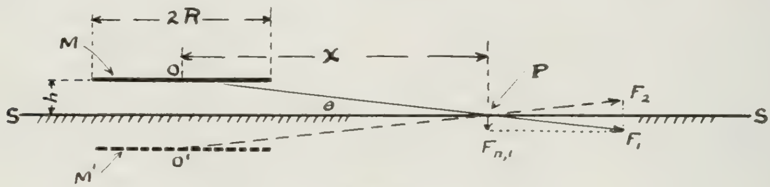

FIG. 1

2. THE ELECTROSTATIC FIELD AT A GREAT DISTANCE FROM AN EXTENDED CHARGED SHEET

In Fig. 1, let M represent an extended circular sheet of conducting material insulated from and parallel to the surface of the earth. Imagine this conducting or capacity area M to be maintained at a steady voltage E above the potential of the earth, and let us calculate the potential gradient which is thereby set up at a point P near the surface of the earth and at a great distance from the capacity area.

Let E represent the difference of potential in volts between the area M and the earth.
Let x represent the horizontal distance from the center of M to the point P.
Let R represent the radius of the area M.
Let h represent the height of M above the earth.
Let p represent the permittivity of the air (8.84×10^{-14})
All distances are to be expressed in centimeters.

Suppose now that h is small in comparison with R and that the distance x in comparison with R is very large. For

example, suppose the radius R is of the order of 60 meters (200 ft.), that the height h is between 1 meter and 30 meters (3 and 100 feet), and that the distance x to the point P is 10 kilometers (6.2 miles) or more. For these proportions the following statements are approximately correct.

Neglecting "edge effects" and displacement from the upper face of M, the capacity C of the sheet M with reference to the earth is

$$C = p\frac{\pi R^2}{h} \text{farads}$$

The quantity of electricity (Q) on M is

$$Q = CE = p\frac{\pi R^2}{h} E \text{ coulombs}$$

The potential gradient F_1 at P due to the charge Q on M is exerted in the direction OP and is

$$F_1 = \frac{1}{4\pi p}\ \frac{Q}{(h^2+x^2)} = \frac{R^2E}{4h(h^2+x^2)} \text{ volts per cm.}$$

Since $h^2 < .00001\ x^2$,

$$F_1 = \frac{R^2E}{4hx^2} \text{ volts per cm. (approximately)}$$

The gradient at P which results from the distribution of the charge $(-Q)$ on the surface of the earth is calculated by introducing the electrical image of the charge Q with reference to the equipotential surface SS, which is hereinafter treated as a plane surface. The image of Q is the charge $(-Q)$ located on the surface M′ at the distance h below the plane SS.

The charge $(-Q)$ at M′ would give rise at the point P to a gradient F_2 exerted along the line O′P and equal to

$$F_2 = -\frac{R^2E}{4hx^2} \text{ volts per cm.}$$

The gradients F_1 and F_2 at P may be resolved into components parallel to and normal to the surface of the earth. The components parallel to the surface neutralize, and those

normal to the surface add. The component of F_1 normal to the surface of the earth is—

$$F_{n1} = F_1 \sin \theta = F_1 \frac{h}{\sqrt{h^2 + x^2}} = F_1 \frac{h}{x}$$

$$F_{n1} = \frac{R^2E}{4x^3} \text{ volts per cm.}$$

Therefore the resultant potential gradient F at the point P is normal to the surface of the plane SS, is directed downward, and is approximately expressed by

$$F = \frac{R^2E}{2x^3} \text{ volts per cm.}$$

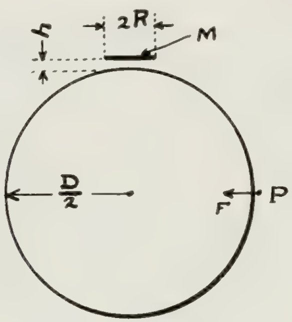

FIG 2

The expression for the gradient at P does not involve h, the height of capacity area M. Therefore, *within the limits previously specified,* the gradient at P due to an extended sheet M maintained at a given potential E above the ground is independent of the height of the sheet above the ground.*

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* It is to be recognized that this treatment is not rigorous and that the conclusions apply only within certain limits. If h is made large as compared with R, the capacity of the sheet M to the plane SS becomes independent of h and equal to 8pR farads. Therefore the gradient at P is normal to the surface of the plane and is equal to $\frac{4Rh\ E}{\pi x^3}$ volts per cm. That is, the gradient at P will be directly porportional to the height of the sheet M above the surface SS.

The fact that the earth's surface is a spherical surface and not a plane surface does not alter the conclusion that the gradient at the point P is independent of the height of the sheet above the ground. For example, by applying the method of images to the spherical surface shown in Fig. 2, it may be shown that the gradient at the point P which is at a quadrant's distance from M is given by the expression

$$F = \frac{\sqrt{2}\ R^2\ E}{D^3} \text{ volts per cm. (approximately)}$$

This expression does not involve h, the height of the sheet M above the surface of the sphere. (As in the case of the plane surface, this expression applies only for the case in which h is small in comparison with R.)

In view of the fact that the height of the capacity area is (within limits) without influence upon the steady state of the medium at P, this question now arises. If the potential of the sheet M, instead of being maintained constant, is caused to vary in a periodic manner, how will the magnitude of the disturbance thereby set up in the medium at the point P be affected by the height of the sheet M?

3. Simplification of the Radiating System

To answer the question thus raised by the discussion of the steady state of the field requires the application of the equations of the electro magnetic field to the radiating system shown in Fig. 3. In its simplest form the radiating system

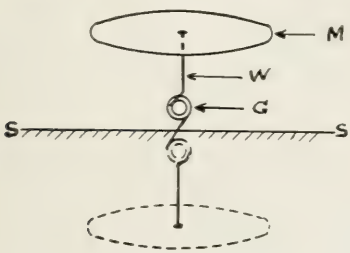

Fig. 4

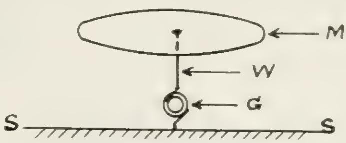

Fig. 3

comprises an extended circular plate M with its surface parallel to the surface of the earth, a vertical conductor W and a generator G generating a sine e. m. f. To treat such a system, the surface SS is imagined to have infinite conductivity. It thus becomes an equipotential surface and the effect of the distributions of current and charge over this surface is determined by replacing this conducting surface by the images of elements M, W, and G in the surface, as in Fig. 4.

The conditions to be fulfilled at the boundaries of Fig. 4—that is, at the surfaces of the conductors—are so involved that a rigorous analytical treatment is impossible. If the conditions are simplified by assuming the conducting sheet M and wire W to have infinite conductivity, the boundary conditions are still too involved for treatment. It becomes necessary, therefore, to further simplify the radiating system by depicting it as in Fig. 5. In Fig. 5 a positive charge Q distributed over a circular area of radius R is assumed to

move up and down in such a manner that at any moment its elevation (h) above the surface SS is given by the expression

$$h = h_0 \cos \omega t$$

A negative charge ($-Q$), also distributed over a circular area of radius R, moves up and down so that its elevation is given by the expression

$$h = -h_0 \cos \omega t$$

The charges Q and $-Q$ are not uniformly distributed over the two circular areas, but they may be imagined to be confined

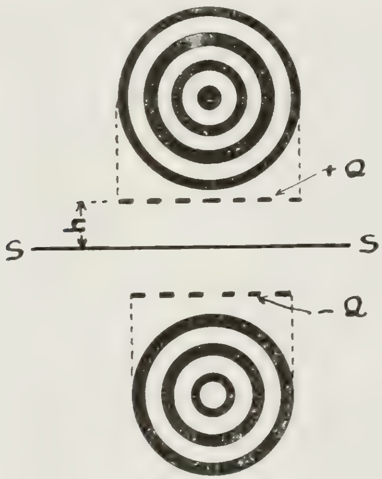

FIG. 5

to insulated circular strips. When the two charges move up and down, the circular strips carrying the positive charge may be imagined to pass between the circular strips carrying the negative charge at the instant both charges pass in opposite directions across the surface SS. The steady states at great distances which correspond to the instantaneous states of such a radiating system are identically the same as the states corresponding to a Fig. 4 system, provided the total voltage generated in the two generators of Fig. 4 is given by the expression

$$e = 2\,E \cos \omega t = \frac{2Q\,h_0}{\pi R^2 p} \cos \omega t$$

In the case of the radiating system depicted in Fig. 5, there are no conduction currents to deal with, and the electric charges move in a simple predetermined manner. We proceed (a) to set up the differential equations applying to this system, (b) to indicate a solution of these equations, and (c) from this solution to draw conclusions as to the relative merits of high versus low capacity areas in wireless telegraphy.

II. DIFFERENTIAL EQUATIONS OF THE ELECTROMAGNETIC FIELD..

4. Notation.

At any point in the electromagnetic field,

Let

ρ represent the volume density of electricity in coulombs per cu. cm.

V " the velocity of the moving charge in cm. per sec.

F " the electric force or potential gradient in volts per cm.

H " the magnetic force in ampere-turns per cm.

D " the electrostatic flux density or displacement in coulombs per sq. cm.

B " the magnetic flux density in webers per sq. cm.

Φ " the retarded displacement potential.

A " the retarded vector potentiial.

Let

f represent the frequency of the radiating system in cycles per sec.

p " the permittivity of the medium in coulombs per sq. cm. per volt per cm.

For free space $p = \frac{1}{4\pi 9\ 10^{11}}$ or $8.84\ 10^{-14}$

μ " the permeability of the medium in webers per sq. cm. per ampere-turn per cm.

For free space $\mu = \frac{4\pi}{10^9}$ or $1.257\ 10^{-8}$

s " $\frac{1}{\sqrt{\mu p}} = 3\ 10^{10}$ = velocity of light.

∇^2 " $\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)$

The quantities represented by bold faced capitals are vector quantities. Let their X, Y, and Z components be designated by the subscripts $_1$, $_2$, and $_3$. Thus,

Let

D_1 D_2 D_3 represent the X, Y, and Z components of the displacement.

V_1 V_2 V_3 represent the X, Y, and Z components of the velocity, etc.

It will be noted that all quantities are to be expressed in terms of the Ampere, Ohm, Ampere-turn, Weber system of units.

5. Fundamental Relations Expressed in Vector Notation

The fundamental relations which must be satisfied at all points of the electromagnetic field are expressed by the following differential equations.

$$\text{div}\ \mathbf{D} = \rho \qquad (1)$$

$$\text{div}\ \mathbf{B} = 0 \qquad (2)$$

$$\text{curl}\ \mathbf{H} = \frac{d\mathbf{D}}{dt} + \rho\mathbf{V} \qquad (3)$$

$$\text{curl}\ \mathbf{F} = -\frac{d\mathbf{B}}{dt} \qquad (4)$$

$$\mathbf{D} = p\mathbf{F} \qquad (5)$$

$$\mathbf{B} = \mu\mathbf{H} \qquad (6)$$

These same relations when expressed by differential equations involving the rectangular components of the vectors take the following forms.

6. Fundamental Relations Expressed in Rectangular Coordinates

Equations (1) and (2) may be written:

$$\frac{dD_1}{dx} + \frac{dD_2}{dy} + \frac{dD_3}{dz} = \rho \qquad (1a)$$

$$\frac{dB_1}{dx} + \frac{dB_2}{dy} + \frac{dB_3}{dz} = 0 \qquad (2a)$$

Equation (3) may be written:

$$\mathbf{i}\left(\frac{dH_3}{dy} - \frac{dH_2}{dz}\right) + \mathbf{j}\left(\frac{dH_1}{dz} - \frac{dH_3}{dx}\right) + \mathbf{k}\left(\frac{dH_2}{dx} - \frac{dH_1}{dy}\right) =$$

$$\mathbf{i}\left(\frac{dD_1}{dt} + \rho V_1\right) + \mathbf{j}\left(\frac{dD_2}{dt} + \rho V_2\right) + \mathbf{k}\left(\frac{dD_3}{dt} + \rho V_3\right)$$

This yields the three equations:

$$\frac{dH_3}{dy} - \frac{dH_2}{dz} = \frac{dD_1}{dt} + \rho V_1 \qquad (3a)$$

$$\frac{dH_1}{dz} - \frac{dH_3}{dx} = \frac{dD_2}{dt} + \rho V_2 \qquad (3b)$$

$$\frac{dH_2}{dx} - \frac{dH_1}{dy} = \frac{dD_3}{dt} + \rho V_3 \qquad (3c)$$

In like manner, equation (4) yields the following three equations:

$$\frac{dF_3}{dy} - \frac{dF_2}{dz} = -\frac{dB_1}{dt} \qquad (4a)$$

$$\frac{dF_1}{dz} - \frac{dF_3}{dx} = -\frac{dB_2}{dt} \qquad (4b)$$

$$\frac{dF_2}{dx} - \frac{dF_1}{dy} = -\frac{dB_3}{dt} \qquad (4c)$$

7. Physical Interpretation of the Differential Equations

The relations expressed in the differential equations 1 to 4 are more familiar to engineers when stated in a form more suitable for application to circuits of finite dimensions. It

may, therefore, be well, before we proceed to the solution of these differential equations, to identify the equations with the more familiar statements of the laws they express.

Equation 4 results from the application of Faraday's Law of Induction to a circuit of infinitesimal dimensions. The law of induction is, "The electromotive force induced in a closed circuit, or the line integral of the electric force around the circuit, is equal to the rate of decrease of the magnetic flux threading the circuit." This is called by Heaviside the second law of circuitation. Consider the application of this law to the small circuit bounding the infinitesimal square parallel to the XY plane in Fig. 6.

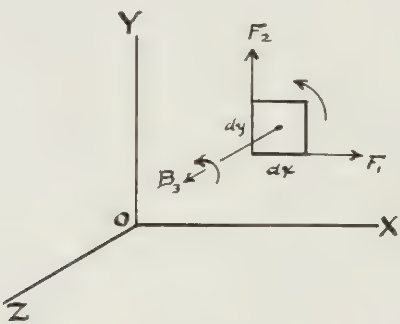

FIG. 6

The magnetic flux threading this circuit is B_3dxdy

The rate of decrease of this flux is $-\frac{dB_3}{dt}$ dxdy

Now if F_1 and F_2 represent the X and Y components of the electric force or voltage gradient at this point, it is evident that the resultant or net electromotive force around the circuit in the direction indicated (or, in other words, the line integral of the electric force around the boundary of the square) is given by the expression

$$\text{Line integral of } \mathbf{F} = F_1\,dx + \left[F_2 + \frac{dF_2}{dx}dx\right]dy - \left[F_1 + \frac{dF_1}{dy}dy\right]dx - F_2dy = \left(\frac{dF_2}{dx} - \frac{dF_1}{dy}\right)dx\,dy$$

According to Faraday's law, the electromotive force induced in a circuit around the boundary of this square is equal to the rate of decrease of the flux threading the square. Therefore the line integral of the electric force **F** around the area dydx or—

$$\left(\frac{dF_2}{dx} - \frac{dF_1}{dy}\right) dx\ dy \text{ is equal to } - \frac{dB_3}{dt} dx\ dy$$

Whence

$$\frac{\text{Line integral of } \mathbf{F} \text{ around } dx\,dy}{\text{area}} \text{ or } \left(\frac{dF_2}{dx} - \frac{dF_1}{dy}\right) = - \frac{dB_3}{dt}$$

This is equation (4c). In like manner equations (4a) and (4b) may be derived.

Now the *curl of a vector* **F** at any point P *and in any plane* passing through that point is defined as a vector **M** whose length is equal to the line integral of the vector **F** taken around the boundary of an infinitesimal portion of the plane divided by the area of the infinitesimal portion. The vector **M** is to be drawn normal to the plane and in that direction in which a right hand screw would advance if it were threaded through the plane and rotated in the direction in which the boundary was traversed in taking the line integral. At the given point P there will be some plane for which this quotient, or the curl, has a maximum value. This maximum value is called "*The* Curl of the vector **F** at the point P." If the curl of the vector **F** in three planes parallel to the XY, XZ, and YZ planes is taken, the three vectors so obtained are the Z, Y, and X components of "*The* Curl of the vector."

To summarize the above discussion, the quotient obtained by dividing the line integral of the electric force **F** around the boundary of a small area parallel to the XY plane by the area was found to be $\left(\frac{dF_2}{dx} - \frac{dF_1}{dy}\right)$. Through the application of Faraday's Law of Induction, this quotient was shown to equal the rate of decrease of the Z component of the flux density at the point. In other words, the curl of the electric force **F** in a plane parallel to the XY plane, or the Z com-

ponent of the curl of $\mathbf{F}$, is equal to the rate of decrease of the Z component of the flux density. Likewise, the X and Y components of the curl of $\mathbf{F}$ are equal respectively to the rates of decrease of the X and Y components of the flux density B. Or

$$\text{curl}\,\mathbf{F} = -\frac{d\mathbf{B}}{dt} \tag{4}$$

8. Equation (3) is obtained by applying to a circuit of infinitesimal dimensions the familiar conception that the

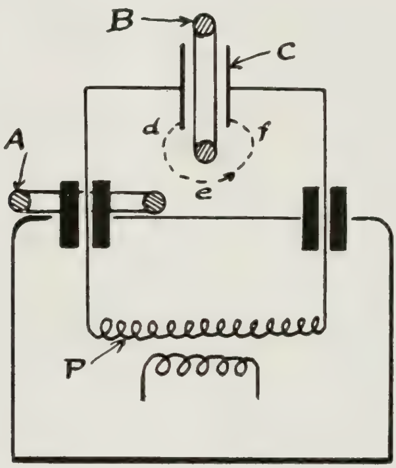

Fig. 7

magnetomotive force in ampere turns exerted around any complete circuit, or the line integral of the magnetic force around the circuit, is equal to the current passing through and looping with the circuit around which the line integral is taken. By the current passing through the circuit is meant the sum of the conduction current plus the convection current plus the displacement current. This is called by Heaviside the first law of circuitation.

To illustrate a specific application of this law to a circuit of finite dimensions, suppose we wish to determine the magnetomotive force exerted upon the magnetic circuit—the iron core—of the current transformer illustrated in two different positions A and B in Fig. 7.

To find the current passing through and looped with the core we imagine any surface, plane or curved, of which the core is the boundary. The current looping with the iron core at any instant is, then, the net current passing across this surface. Imagine a plane surface of which the core is the boundary. Then with the current transformer in the position A, substantially the only current which crosses the plane surface is the conduction current in the high tension lead of the power transformer P. (The secondary circuit of the current transformer is assumed to be open.)

Suppose, however, the current transformer is shifted to the position B, a position in which the plane surface bounded by the core cuts through the dielectric of the condenser C, and consequently, a position in which no conduction current crosses the plane surface. In this position the current crossing the plane surface is a displacement current. This displacement current—the rate of change of the electrotrostatic flux which passes across the plane surface bounded by the core—is less than the conduction current in the transformer lead at A by the displacement which takes place between the leads along paths, as *d e f*, which do not loop through the iron core. If the leads are short and the condenser plates large, the magnetomotive force exerted upon the core in the position B will be only slightly lower than in the position A.

As in the previous case, let these considerations be applied to a small square circuit of infinitesimal dimensions similar to that shown in Fig. 6.

The line integral of the magnetic force H around the boundary of the square is—

$$\left(\frac{dH_2}{dx} - \frac{dH_1}{dy}\right) dx\ dy$$

The current passing through the area dxdy is—

$$\left(\rho V_2 + \frac{dD_3}{dt}\right) dx\ dy$$

Since the magnetomotive force around the boundary of the area equals the current through the area, these expressions are equal.

$$\text{Whence} \left(\frac{dH_2}{dx} - \frac{dH_1}{dy}\right) dx\, dy = \left(\rho V_3 + \frac{dD_3}{dt}\right) dx\, dy$$

$$\text{or} \left(\frac{dH_2}{dx} - \frac{dH_1}{dy}\right) = \left(\rho V_3 + \frac{dD_3}{dt}\right)$$

This is equation (3c). The left member is the curl of the magnetic force **H** in a plane parallel to the XY plane, or it is the Z component of the curl of **H**. The right member is the expression for the Z component of the convection current density plus the Z component of the displacement current density.

Equations (3a) and (3b) may be derived in a similar manner.

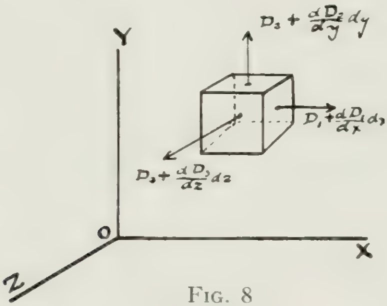

FIG. 8

9. Equation (1) expresses the fact that the Faraday tubes of electric force originate on the electric charges. The relation is perhaps more familiar in either of the following forms: "The number of tubes of displacement which cross any closed surface in the field is equal to the quantity of electricity contained within the surface," or, "The surface integral of the displacement taken over any closed surface is equal to the quantity of electricity contained within the surface."

Let this law be applied to the small cubical volume shown in Fig. 8.

The surface integral of the displacement taken over the surface of the cube is:

$$-D_1\, dydz + \left[D_1 + \frac{dD_1}{dx}dx\right] dy\, dz - D_2\, dx\, dz$$

$$+\left[D_2 + \frac{dD_2}{dy}dy\right] dxdz - D_3\, dxdy + \left[D_3 + \frac{dD_3}{dz}dz\right] dxdy$$

$$\text{or} \left[\frac{dD_1}{dx} + \frac{dD_2}{dy} + \frac{dD_3}{dz}\right] dxdydz$$

The quantity of electricity within the cube is ρ dxdydz

$$\text{Whence} \left[\frac{dD_1}{dx} + \frac{dD_2}{dy} + \frac{dD_3}{dz}\right] dxdydz = \rho\, dxdydz$$

$$\text{or} \left[\frac{dD_1}{dx} + \frac{dD_2}{dy} + \frac{dD_3}{dz}\right] = \rho \tag{1a}$$

The left member of equation (1a) is the quotient obtained by dividing the surface integral of D taken over the surface of the infinitesimal cube by the volume of the cube. The value of this quotient is called the divergence of the vector D at the point.

10. Equation 2 expresses the fact that the tubes of magnetic induction do not originate or diverge from any portion of space. They are closed tubes, linking with the current and returning into themselves.

III. INTEGRATION OF THE DIFFERENTIAL EQUATIONS

11. To Obtain the Differential Equations in a Form Involving Only One Dependent Variable, as D_1

Differentiating (3a) with respect to (t),

$$\frac{d^2 H_3}{dt\, dy} - \frac{d^2 H_2}{dt\, dz} = \frac{d^2 D_1}{dt^2} + \frac{d}{dt}(\rho V_1) \tag{7}$$

Substituting in (4b) and (4c) μH for B, differentiating (4c) with respect to y, and (4b) with respect to z, and substituting the values so obtained for $\frac{d^2 H_3}{dt\, dy}$ and $\frac{d^2 H_2}{dt\, dz}$ in equation (7),

$$\frac{1}{\mu}\left[\frac{d^2 F_1}{dy^2} - \frac{d^2 D_2}{dy\, dx} + \frac{d^2 F_1}{dz^2} - \frac{d^2 F_3}{dz\, dx}\right] = \frac{d^2 D_1}{dt^2} + \frac{d}{dt}(\rho V_1) \quad 8$$

Substituting for F in equation (8), its value from equation (5),

$$\frac{1}{\mu p}\left[\frac{d^2 D_1}{dy^2} - \frac{d^2 D_2}{dy\, dx} + \frac{d^2 D_1}{dz^2} - \frac{d^2 D_3}{dz\, dx}\right] = \frac{d^2 D_1}{dt^2} + \frac{d}{dt}(\rho V_1) \quad 9$$

Differentiating (1a) with respect to x and substituting the value so obtained for $\left(-\frac{d^2 D_2}{dy\, dx} - \frac{d^2 D_3}{dz\, dx}\right)$ in equation (9)

$$\left[\frac{d^2 D_1}{dx^2} + \frac{d^2 D_1}{dy^2} + \frac{d^2 D_1}{dz^2}\right] - \mu p \frac{d^2 D_1}{dt^2} = \frac{d\rho}{dx} + \mu p \frac{d}{dt}(\rho V_1) \quad (10a)$$

Writing s for $\frac{1}{\sqrt{\mu p}} = 3\ 10^{10}$

and ∇^2 for $\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)$

Equation (10a) may be written,

$$\nabla^2 D_1 - \frac{1}{s^2}\frac{d^2 D_1}{dt^2} = \frac{d\rho}{dx} + \frac{1}{s^2}\frac{d}{dt}(\rho V_1) \quad (10a)$$

$$\text{or } \left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right] D_1 = \frac{d\rho}{dx} + \frac{1}{s^2}\frac{d}{dt}(\rho V_1) \quad (10a)$$

In like manner the following equations may be obtained,

$$\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right] D_1 = \frac{d\rho}{dy} + \frac{1}{s^2}\frac{d}{dt}(\rho V_1) \quad (10b)$$

$$\text{and } \left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right] D_3 = \frac{d\rho}{dz} + \frac{1}{s^2}\frac{d}{dt}(\rho V_3) \quad (10c)$$

12. To Obtain Differential Equations Involving H_1 Only

Differentiating (4a) with respect to (t), and substituting for B_1 its value μH_1,

$$\frac{d^2 F_3}{dt\, dy} - \frac{d^2 F_2}{dt\, dz} = -\mu \frac{d^2 H_1}{dt^2} \quad (11)$$

Substituting for D in equations (3b) and (3c) its value pF, differentiating (3c) with respect to (y) and (3b) with respect to (z), and substituting the values so obtained for $\frac{d^2 F_3}{dt\, dy}$ and $\frac{d^2 F_2}{dt\, dz}$ in equation (11),

$$\frac{1}{p}\left[\frac{d^2 H_2}{dx\, dy} - \frac{d^2 H_1}{dy^2} - \frac{d}{dy}(\rho V_3) - \frac{d^2 H_1}{dz^2} + \frac{d^2 H_3}{dz\, dx} + \frac{d}{dz}(\rho V_2)\right]$$

$$= -\mu \frac{d^2 H_1}{dt^2} \quad (12)$$

Differentiating (2a) with respect to (x) and substituting the value so obtained for $\left(\frac{d^2 H_2}{dx\,dy} + \frac{d^2 H_3}{dz\,dx}\right)$ in equation (12),

$$\left[\frac{d^2 H_1}{dx^2} + \frac{d^2 H_1}{dy^2} + \frac{d^2 H_1}{dz^2}\right] - \mu p \frac{d^2 H_1}{dt^2} = -\left[\frac{d}{dy}(\rho V_3) - \frac{d}{dz}(\rho V_2)\right] \qquad (13a)$$

This may be written:

$$\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right] H_1 = -\left[\frac{d}{dy}(\rho V_3) - \frac{d}{dz}(\rho V_2)\right] \qquad (13a)$$

In like manner the following equations may be obtained:

$$\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right] H_2 = -\left[\frac{d}{dz}(\rho V_1) - \frac{d}{dx}(\rho V_3)\right] \qquad (13b)$$

and $$\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right] H_3 = -\left[\frac{d}{dx}(\rho V_2) - \frac{d}{dy}(\rho V_1)\right] \qquad (13c)$$

13. The Dalembertian Operator

The differential equations for which the solution is desired take the forms:

$$\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right] D_1 = \frac{d\rho}{dx} + \frac{1}{s^2}\frac{d}{dt}(\rho V_1) \qquad (10a)$$

and $$\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right] H_1 = -\left[\frac{d}{dy}(\rho V_3) - \frac{d}{dz}(\rho V_2)\right] \qquad (13a)$$

That is to say, in the solutions or integrated equations, D_1 and H_1 must be such functions of time and of the position of a point in space that the operation $\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right]$ applied to the function will yield a result whose value is determined by the volume density (ρ) and the current density ($\rho \mathbf{V}$) at the point.

It is very convenient to have a name for the result of the operation $\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right]$ upon a quantity. H. A. Lorentz has suggested* that the result of the operation be

* H. A. Lorentz, *Theory of Electrons*, page 17.

called the "Dalembertian of the quantity," since d'Alembert was the first to solve the differential wave equation involving the operation $\left[\frac{d^2}{dx^2} - \frac{1}{s^2}\frac{d^2}{dt^2}\right]$ which is a special case of $\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right]$.

Adopting this suggestion, we may say that the Dalembertians of the components of $\mathbf{D}$ and $\mathbf{H}$ are given in equations (10) and (13) in terms of the volume density (ρ) and current density ($\rho\mathbf{V}$).

Now in the system of moving charges depicted in Fig. 5, (ρ) and ($\rho\mathbf{V}$) are known functions of time and of the position of a point in space. That is to say, the right hand members of equations (10a) and (13a) are functions of known form, and the problem is to find the functional expression which will give the value of D_1, or of H_1, at any point in space and for any moment of time. If the known form assumed by the right hand member of equation (10a) is represented by the expression f (t, X, Y, Z), we have given—

$$\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right]D_1 = f\,(t, X, Y, Z) \qquad (14)$$

Or, dropping the subscripts, the problem is—

Given, Dalembertian D = f (t, X, Y, Z),

to find, the expression for D.

We proceed to demonstrate the following proposition.

14. Proposition That the "Retarded Potentializing Operation" Performed Upon the f (t, x, y, z) Yields a Function Whose Dalembertian Equals the f (t, x, y, z).*

"If the Dalembertian of D equals f (t, X, Y, Z), then the value of D may be found by an operation which may be called the operation of forming the "retarded potential" of the f (t, X, Y, Z). The operation of forming the retarded potential of the f (t, X, Y, Z) may be symbolized by stating that the value of D will be given by the equation,

* Whittaker, *History of the Theories of Aether and Electricity*, pages 268 & 298.
H. A. Lorentz, *The Theory of Electrons*, page 233.
Ludwig Lorenz, *Phil. Mag.* (1867) Vol. 34, page 287.
M. B. Riemann, *Phil. Mag.* (1867) Vol. 34, page 368.

$$D = -\frac{1}{4\pi}\int^{\text{all space}} \frac{f\left[\left(t-\frac{r}{s}\right), X, Y, Z,\right]}{r}\,dv \qquad (15)$$

This expression is to be read as follows: The value of D at a given instant (t) and for a given point P ≡ (x, y, z) is equal to $\left(-\frac{1}{4\pi}\right)$ times the summation obtained—

(a) by dividing all space into volume elements (dv),

(b) dividing the volume of each element by its distance $r = \sqrt{(X-x)^2 + (Y-y)^2 + (Z-z)^2}$ from the point P.

(c) multiplying this quotient by the value of the f (t, X, Y, Z) at the volume element, *not for the instant* (*t*), but for the instant of time $\left(t - \frac{r}{s}\right)$, or r/s seconds earlier.

(d) and finally summing up all the products so obtained."

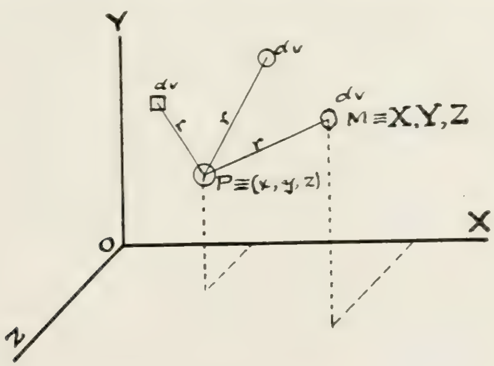

Fig. 9

Before taking up the proof of this proposition it may be noted that if there are no moving charges in the field the operation of obtaining the retarded potential becomes identically the same as the more familiar operation of obtaining the potentials in the gravitational or in the electrostatic field. If there are no moving charges, the Dalembertian of D degrades to the Laplacian of D.

15. Proof for Charges at a Distance from the Point

To show that the value of D defined by equation (15) satisfies the differential relation expressed in equation (14).

Let $P \equiv (x, y, z)$, Fig. 9, be any point for which the value of D is desired for the instant (t). In performing the integration indicated in equation (15), we are concerned with only those volume elements for which the f (t, X, Y, Z) has a value at the instant $\left(t - \frac{r}{s}\right)$. That is, only those volume elements which contain a charge at the instant $\left(t - \frac{r}{s}\right)$ contribute to the integral. All the balance of space contributes nothing to the integral. A few of the volume elements presumed to contribute to the integral have been shown in Fig. 9.

Consider first the part of the integral contributed by any volume element whatsoever except the element immediately surrounding the point P.

The part of the integral contributed by the volume element (dv) at $M \equiv (X, Y, Z)$ is—

$$D_M = -\frac{1}{4\pi}\,\frac{f\left[\left(t - \frac{r}{s}\right), X, Y, Z\right]}{r}\,dv$$

$$= -\frac{1}{4\pi}\,\frac{f\left[\left(t - \frac{\sqrt{(X-x)^2 + (Y-y)^2 + (Z-z)^2}}{s}\right), X, Y, Z\right]}{\sqrt{(X-x)^2 + (Y-y)^2 + (Z-z)^2}}\,dv \qquad (16)$$

Taking the second derivative of D_M with respect to (t),

$$\frac{d^2}{dt^2}D_M = -\frac{1}{4\pi}\,\frac{f''\left[\left(t - \frac{r}{s}\right), X, Y, Z\right]}{r}\,dv \qquad (17)$$

(See the footnote* for the meaning of $f''\,[(t - r/s), X, Y, Z]$)

*In the expression $f\left[\left(t - \frac{r}{s}\right), X, Y, Z\right]$, let $\left(t - \frac{r}{s}\right)$ be represented by (u). Then—

$$\frac{d}{dx}\,f\left[\left(t - \frac{r}{s}\right), X, Y, Z\right] = \frac{du}{dx}\,\frac{d}{du}\,f\left[u, X, Y, Z\right]$$
$$= \frac{du}{dx}\,f'\left[u, X, Y, Z\right]$$

and

$$\frac{d^2}{dx^2}\,f\left[\left(t - \frac{r}{s}\right), X, Y, Z\right] = \frac{d}{dx}\left\{\frac{du}{dx}\,f'\left[u, X, Y, Z\right]\right\}$$
$$= \frac{d^2u}{dx^2}\,f'\left[u, X, Y, Z\right] + \left(\frac{du}{dx}\right)^2 f''\left[u, X, Y, Z\right]$$
$$= \frac{d^2u}{dx^2}\,f'\left[\left(t - \frac{r}{s}\right), X, Y, Z\right] + \left(\frac{du}{dx}\right)^2 f''\left[\left(t - \frac{r}{s}\right), X, Y, Z\right]$$

Taking the second derivative of D_M with respect to (x),

$$\frac{d^2}{dx^2} D_M = -\frac{dv}{4\pi} \left\{ f''\left[\left(t-\frac{r}{s}\right), X,Y,Z\right] \frac{(X-x)^2}{s^2r^3} \right.$$

$$+ f'\left[\left(t-\frac{r}{s}\right), X,Y,Z\right] \left[\frac{3(X-x)^2}{sr^4} - \frac{1}{sr^2}\right]$$

$$\left. + f\left[\left(t-\frac{r}{s}\right), X,Y,Z\right] \left[\frac{3(X-x)^2}{r^5} - \frac{1}{r^3}\right] \right\}$$

In like manner, the following derivatives are obtained:

$$\frac{d^2}{dy^2} D_M = -\frac{dv}{4\pi} \left\{ f''\left[\left(t-\frac{r}{s}\right), X,Y,Z\right] \frac{(Y-y)^2}{s^2r^3} \right.$$

$$+ f'\left[\left(t-\frac{r}{s}\right), X,Y,Z\right] \left[\frac{3(Y-y)^2}{sr^4} - \frac{1}{sr^2}\right]$$

$$\left. + f\left[\left(t-\frac{r}{s}\right), X,Y,Z\right] \left[\frac{3(Y-y)^2}{r^5} - \frac{1}{r^3}\right] \right\}$$

and

$$\frac{d^2}{dz^2} D_M = -\frac{dv}{4\pi} \left\{ f''\left[\left(t-\frac{r}{s}\right), X,Y,Z\right] \frac{(Z-z)^2}{s^2r^3} \right.$$

$$+ f'\left[\left(t-\frac{r}{s}\right), X,Y,Z\right] \left[\frac{3(Z-z)^2}{sr^4} - \frac{1}{sr^2}\right]$$

$$\left. + f\left[\left(t-\frac{r}{s}\right), X,Y,Z\right] \left[\frac{3(Z-z)^2}{r^5} - \frac{1}{r^3}\right] \right\}$$

Adding

$$\left[\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right] D_M = -\frac{1}{4\pi s^2} \frac{f''\left[\left(t-\frac{r}{s}\right), X,Y,Z\right]}{r} dv \tag{18}$$

Hence from equations (17) and (18),

$$\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right] D_M = 0$$

As the volume element at M represents any volume element save the element surrounding P, it follows that the Dalembertian of all that portion of D which is contributed

by volume elements other than the element immediately surrounding P is zero.

There remains to be considered the value contributed to the integral by the volume element N immediately surrounding the point P. Let this element be taken as a small spherical volume of radius R. To establish our proposition, the Dalembertian of the quantity which is contributed to D by this volume element must be demonstrated to equal the value of the f (t, x, y, z) at P.

If the $f\left[\left(t-\frac{r}{s}\right), X,Y,Z\right]$ is zero within the sphere N, this portion of space contributes nothing to D. Therefore, if at P the f (t, x, y, z) is zero, the Dalembertian of D for the point P is zero, and the value obtained for D by the summation expressed by equation (15) satisfies the differential relation expressed in equation (14).

16. Proof for Charges at the Point

If the, $f\left[\left(t-\frac{r}{s}\right), X,Y,Z\right]$ is not zero within the small sphere N surrounding the point P — in other words, if this sphere contains a charge—the value D_N contributed to D by the values of $f\left[\left(t-\frac{r}{s}\right), X,Y,Z\right]$ within this spherical volume must be determined, and, as previously stated, to establish our proposition, the Dalembertian of D_N must be shown to equal the value of the f (t, x, y, z) at the point P.

The value of D_N is defined by the equation,

$$D_N = -\frac{1}{4\pi}\int^{\text{Vol. of the sphere}} \frac{f\left[\left(t-\frac{r}{s}\right), X,Y,Z\right]}{r} dv$$

In the first place, it is to be noted that the infinite value assumed by the integrand when r equals zero does not mean that the integral D_N is infinite. This may be demonstrated as follows:

Over the space within a sphere of infinitesimal radius R the $f\left[\left(t-\frac{r}{s}\right), X,Y,Z\right]$ will, for any given instant of time,

have substantially the same values at all points within the sphere. Or, at any rate, the values of the function for the different points within the sphere may be conceived to lie between a maximum value f_1 and a minimum value f_2. By letting the radius of the sphere approach zero, these values may be made to differ by an infinitesimal amount from the value of the function at the center of the sphere, namely f (t, x, y, z).

Consequently, the $f\left[\left(t-\frac{r}{s}\right), X,Y,Z\right]$ may be imagined to have the uniform value f (t, x, y, z) at all points within the sphere. The value of D_N may then be computed by dividing

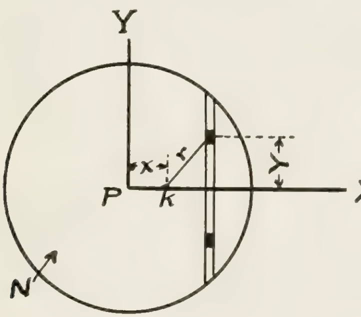

FIG. 10

up the spherical volume into spherical shells of thickness (dr) and carrying out the indicated integration.

Whence

$$D_N = -\frac{1}{4\pi}\int_0^R \frac{f\,(t,x,y,z)\,4\pi\, r^2\, dr}{r}$$

$$= -\frac{R^2}{2}\, f\,(t,x,y,z) \qquad (19)$$

The value of D_N is therefore not only finite, but it can be caused to decrease without limit by decreasing the radius of the sphere without limit.

For the purpose of calculating the Dalembertian of the value D_N which is contributed to the retarded potential by the space within the sphere N, let the origin of the system

of coordinates be transferred to the center (P) of the sphere. Then let the retarded potential be calculated—not for the point P—but for a point K, Fig. 10, displaced along the X axis by an infinitesimal distance (x) from the center. (This is for the purpose of obtaining the expression for D_N in such a form that the value of $\frac{d^2 D_N}{dx^2}$ may be calculated.)

Imagine the volume of the sphere N to be made up of plane slices of thickness (dX), and these slices in turn to be constituted of circular rings of radius (Y), as shown in Fig. 10.

The expression for the integral D_N now takes the form:

$$D_N = -\frac{1}{4\pi}\int_{X=-R}^{X=x}\int_{Y=o}^{Y=\sqrt{R^2-X^2}} \frac{f\,(t,x,y,z)\;2\pi\,YdYdX}{\sqrt{(x-X)^2+Y^2}}$$

$$-\frac{1}{4\pi}\int_{X=x}^{X=R}\int_{Y=o}^{Y=\sqrt{R^2-X^2}} \frac{f\,(t,x,y,z)\;2\pi\,YdYdX}{\sqrt{(X-x)^2+Y^2}} \qquad (20)$$

Integrating,

$$D_N = -\left(\frac{R^2}{2}-\frac{x^2}{6}\right) f\,(t,x,y,z) \qquad (21)$$

Taking the second derivative of D_N with respect to x and letting R approach zero, the second derivative approaches the following limit:

$$\frac{d^2D_N}{dx^2} = \frac{1}{3}\, f\,(t,x,y,z)$$

In like manner it may be shown that

$$\frac{d^2D_N}{dy^2} = \frac{1}{3}\, f\,(t,x,y,z)$$

and $$\frac{d^2D_N}{dz^2} = \frac{1}{3}\, f\,(t,x,y,z)$$

Taking the second derivative of (19) or (21) with respect to (t) and letting R approach zero, the second derivative approaches the following limit.

$$\frac{d^2 D_N}{dt^2} = 0$$

Therefore

$$\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right] D_N = f\ (t,x,y,z)$$

This establishes the proposition that the values of D determined by equation (15) satisfy the differential relation expressed in equation (14), or the values of a time and point function D whose Dalembertian is a given time and point function f (t, x, y, z) may be found by the operation of forming the retarded potential of the f (t, x, y, z).

17. Proof that the Forces May be Calculated From the "Retarded Displacement Potential" and the "Retarded Vector Potential."

By equations (10a) and (15),

$$D_1 = -\frac{1}{4\pi}\int \frac{\left[\frac{d\rho}{dx} + \frac{1}{s^2}\frac{d}{dt}(\rho V_1)\right]_{(t-r/s)}}{r}\, dv \quad * \qquad (22)$$

Splitting the right hand member of (22) into two parts, equation (22) may be written

$$D_1 = D_1' + D_1'' = -\frac{1}{4\pi}\int \frac{\left[\frac{d\rho}{dx}\right]_{(t-r/s)}}{r}\, dv$$

$$-\frac{1}{4\pi}\int \frac{\left[\frac{1}{s^2}\frac{d}{dt}(\rho V_1)\right]_{(t-r/s)}}{r}\, dv \qquad (22a)$$

In which, D_1' represents the 1st, and D_1'' the 2nd term.

* For the meaning of the subscript (t − r/s), see the footnote to equation (26).

In like manner,

$$D_2 = D_2' + D_2'' = -\frac{1}{4\pi}\int\frac{\left[\frac{d\rho}{dy}\right]_{(t-r/s)}}{r}dv$$

$$-\frac{1}{4\pi}\int\frac{\left[\frac{1}{s^2}\frac{d}{dt}(\rho V_2)\right]_{(t-r/s)}}{r}dv \qquad (22b)$$

$$D_3 = D_3' + D_3'' = -\frac{1}{4\pi}\int\frac{\left[\frac{d\rho}{dz}\right]_{(t-r/s)}}{r}dv$$

$$-\frac{1}{4\pi}\int\frac{\left[\frac{1}{s^2}\frac{d}{dt}(\rho V_3)\right]_{(t-r/s)}}{r}dv \qquad (22c)$$

By equations (13) and (15),

$$H_1 = -\frac{1}{4\pi}\int\frac{-\left[\frac{d}{dy}(\rho V_3) - \frac{d}{dz}(\rho V_2)\right]_{(t-r/s)}}{r}dv \qquad (23a)$$

$$H_2 = -\frac{1}{4\pi}\int\frac{-\left[\frac{d}{dz}(\rho V_1) - \frac{d}{dx}(\rho V_3)\right]_{(t-r/s)}}{r}dv \qquad (23b)$$

$$H_3 = -\frac{1}{4\pi}\int\frac{-\left[\frac{d}{dx}(\rho V_2) - \frac{d}{dy}(\rho V_1)\right]_{(t-r/s)}}{r}dv \qquad (23c)$$

The right hand members of equations (22) and (23) contain terms of the type $\frac{d\rho}{dx}$ and $\frac{d}{dy}(\rho V_3)$. Now the values of these terms depend upon the actual space distribution of the moving charge. Thus far, we have not specified in detail the manner in which the moving charges of Fig. 5 are distributed. All that has been stated of Fig. 5 is that two charges, a positive charge Q and a negative charge -Q, are distributed in some manner over two sets of thin circular rings which move up and down in a simple harmonic manner. The values of D_1, D_2, D_3 and of H_1, H_2, H_3 might be computed by making any arbitrary assumptions as to the distribution of the charges over the rings, subject only to the condition that the total charge is to equal $+Q$ on one set and $-Q$ on the other set of rings. It is far more convenient,

however, to follow a procedure which involves no specific assumptions of this kind. This procedure is as follows:

The components of the displacement and of the magnetic force may be derived from two auxiliary point functions defined as follows:

Let the "retarded displacement potential*" Φ be defined as a scalar point function satisfying the differential relation,

$$\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right]\Phi = -\rho \qquad (24)$$

Also let the "retarded vector potential" $\mathbf{A}$ be defined as a vector point function satisfying the differential relation,

$$\left[\nabla^2 - \frac{1}{s^2}\frac{d^2}{dt^2}\right]\mathbf{A} = -\rho\mathbf{V} \qquad (25)$$

In other words, Φ and $\mathbf{A}$ are defined as quantities whose Dalembertians are to be equal to the negative of the volume density and the negative of the current density respectively. Therefore the values of Φ and $\mathbf{A}$ will be found by the operation of forming the retarded potentials of $-\rho$ and $-\rho\mathbf{V}$. Thus

$$\Phi = \frac{1}{4\pi}\int\frac{\left[\rho\right]_{(t-r/s)}{}^{\dagger}}{r}dv \qquad (26)$$

$$\mathbf{A} = \frac{1}{4\pi}\int\frac{\left[\rho\mathbf{V}\right]_{(t-r/s)}}{r}dv \qquad (27)$$

The components of the vector $\mathbf{A}$ will be given by the equations,

$$A_1 = \frac{1}{4\pi}\int\frac{\left[\rho V_1\right]_{(t-r/s)}}{r}dv \qquad (27a)$$

$$A_2 = \frac{1}{4\pi}\int\frac{\left[\rho V_2\right]_{(t-r/s)}}{r}dv \qquad (27b)$$

* "The *displacement* potential" should not be confused with "*the* potential." The gradient of the former gives displacement and of the latter, electric force. The value of the former is p (the permittivity) times the latter.

†The subscripts (t—r/s) appearing in equations (26) and (27) indicate that in finding the retarded potential at a point P for the instant of time (t), any element of volume is to be multiplied by the value of ρ or $\rho\mathbf{V}$ in the element at the instant (t—r/s).

$$A_3 = \frac{1}{4\pi}\int \frac{\left[\rho V_3\right]_{(t-r/s)}}{r}\, dv \tag{27c}$$

It may be shown that,

$$D_1' = -\frac{d\Phi}{dx} \text{ and } D_1'' = -\frac{1}{s^2}\frac{d}{dt}A_1 \text{ or } D_1 = -\frac{d\Phi}{dx} - \frac{1}{s^2}\frac{d}{dt}A_1 \tag{28a}$$

$$D_2' = -\frac{d\Phi}{dy} \text{ and } D_2'' = -\frac{1}{s^2}\frac{d}{dt}A_2 \text{ or } D_2 = -\frac{d\Phi}{dy} - \frac{1}{s^2}\frac{d}{dt}A_2 \tag{28b}$$

$$D_3' = -\frac{d\Phi}{dz} \text{ and } D_3'' = -\frac{1}{s^2}\frac{d}{dt}A_3 \text{ or } D_2 = -\frac{d\Phi}{dz} - \frac{1}{s^2}\frac{d}{dt}A_3 \tag{23c}$$

Or in vector notation,

$$\mathbf{D} = -\text{ grad }\Phi - \frac{1}{s^2}\frac{d}{dt}\mathbf{A} \tag{28}$$

And that

$$H_1 = \frac{dA_3}{dy} - \frac{dA_2}{dz} = \text{X component of curl of } \mathbf{A} \tag{29a}$$

$$H_2 = \frac{dA_1}{dz} - \frac{dA_3}{dx} = \text{Y component of curl of } \mathbf{A} \tag{29b}$$

$$H_3 = \frac{dA_2}{dx} - \frac{dA_1}{dy} = \text{Z component of curl of } \mathbf{A} \tag{29c}$$

Or in vector notation,

$$\mathbf{H} = \text{curl } \mathbf{A} \tag{29}$$

The proof of these statements is as follows:

18. To Show That $D_1' = -\frac{d\Phi}{dx}$

Let P, Fig. 11, represent any point in space and K a second point displaced from P in a direction parallel to the X axis by the infinitesimal amount (dx). Let the values of the retarded potential Φ at these two points for a given instant of time be represented by Φ_k and Φ_p.

Let the operation of forming the retarded potentials at the points P and K for the instant (t) be visualized as carried out in the following manner:

First: Visualize all space as divided into volume elements, with radii extending from the point P to all those volume elements which contain a charge at the instant $(t - r/s)$. A few of these volume elements and radii have been shown in Fig. 11. The value of Φ_p is the result of carrying out the summation expressed by equation (26) over this system.

Second: Now visualize a second set of radii (indicated by the dotted lines in Fig. 11) all issuing from the point K and

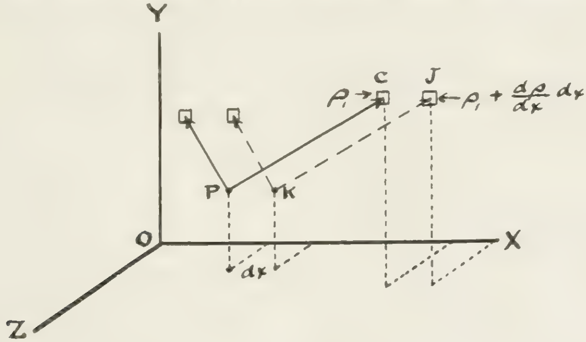

FIG. 11

drawn parallel and equal to the radii issuing from point P. The value Φ_k is the result of carrying out the summation expressed by equation (26) over this system.

How do the summations for Φ_p and Φ_k differ? Every radius with its terminal volume element in one system can be matched by a corresponding radius and volume element in the other. The only difference is in the density of the charge ρ in corresponding volume elements. If the volume density in any element belonging to the point P system (as the element C of Fig. 11) is ρ_1, the volume density in the corresponding element of the point K system (J of Fig. 11) is $\left(\rho_1 + \frac{d\rho}{dx} dx\right)$. Therefore the values of Φ_p and Φ_k will be given by summations involving identical combinations of (r) and (dv) associated with the volume density ρ in the P system and with $\left(\rho + \frac{d\rho}{dx} dx\right)$ in the K system.

3

$$\Phi_K = \frac{1}{4\pi}\int \frac{\left[\rho + \frac{d\rho}{dx}dx\right]_{(t-r\ s)}}{r} dv$$

$$= \frac{1}{4\pi}\int \frac{\left[\rho\right]_{(t-r\ s)}}{r} dv + \frac{1}{4\pi}\int \frac{\left[\frac{d\rho}{dx}dx\right]_{(t-r\ s)}}{r} dv$$

and

$$\Phi_P = \frac{1}{4\pi}\int \frac{\left[\rho\right]_{(t-r\ s)}}{r} dv$$

Therefore

$$\Phi_K - \Phi_P = \frac{1}{4\pi}\int \frac{\left[\frac{d\rho}{dx}dx\right]_{(t-r\ s)}}{r} dv$$

$$\text{Or}\ \frac{\Phi_k - \Phi_p}{dx} = \frac{1}{4\pi}\int \frac{\left[\frac{d\rho}{dx}\right]_{(t-r\ s)}}{r} dv$$

But $\frac{\Phi_k - \Phi_p}{dx}$ is $\frac{d\Phi}{dx}$

and from equation (22a)

$$D_1' = -\frac{1}{4\pi}\int \frac{\left[\frac{d\rho}{dx}\right]_{(t-r\ s)}}{r} dv$$

Therefore

$$D_1' = -\frac{d\Phi}{dx}$$

In like manner it may be shown that

$$D_2' = -\frac{d\Phi}{dy}$$

$$\text{and}\ D_3' = -\frac{d\Phi}{dz}$$

19. To Show That $D_1'' = -\frac{1}{s^2}\frac{d}{dt}A_1$

From equation (27a), the X component A_1 of the retarded vector potential is

$$A_1 = \frac{1}{4\pi}\int \frac{\left[\rho V_1\right]_{(t-r/s)}}{r}\,dv$$

By visualizing the operations involved in finding the values of A_1 at any point P for two instants of time t_1 and $(t_1 + dt)$, it may be seen that

$$\frac{d}{dt}A_1 \text{ or } \frac{d}{dt}\left[\frac{1}{4\pi}\int \frac{\left[\rho V_1\right]_{(t-r/s)}}{r}\,dv\right]$$

$$= \frac{1}{4\pi}\int \frac{\left[\frac{d}{dt}\left(\rho V_1\right)\right]_{(t-r/s)}}{r}\,dv$$

But from equation (22a),

$$D_1'' = -\frac{1}{4\pi}\int \frac{\left[\frac{1}{s^2}\frac{d}{dt}(\rho V_1)\right]_{(t-r/s)}}{r}\,dv$$

Therefore

$$D_1'' = -\frac{1}{s^2}\frac{d}{dt}A_1$$

In like manner it may be seen that

$$D_2'' = -\frac{1}{s^2}\frac{d}{dt}A_2$$

$$\text{and } D_3'' = -\frac{1}{s^2}\frac{d}{dt}A_3$$

20. To Show That $H_1 = \frac{dA_3}{dy} - \frac{dA_2}{dz}$

By equation (27c),

$$A_3 = \frac{1}{4\pi}\int \frac{\left[\rho V_3\right]_{(t-r/s)}}{r}\,dv$$

By visualizing the summations involved in finding the values of A_3 at a point P and at a second point K displaced from P in a direction parallel to the Y axis by the infinitesimal amount (dy), it may be seen that

$$\frac{d}{dy}A = \frac{1}{4\pi}\int \frac{\left[\frac{d}{dy}(\rho V_3)\right]_{(t-r/s)}}{r}\,dv$$

In a similar way it may be seen that

$$\frac{d}{dz}A_2 = \frac{1}{4\pi}\int \frac{\left[\frac{d}{dz}(\rho V_2)\right]_{(t-r/s)}}{r}\,dv$$

But from equation (23a),

$$H_1 = \frac{1}{4\pi}\int \frac{\left[\frac{d}{dy}(\rho V_3) - \frac{d}{dz}(\rho V_2)\right]_{(t-r/s)}}{r}\,dv$$

Therefore

$$H_1 = \frac{dA_3}{dy} - \frac{dA_2}{dz} \qquad (29a)$$

Similar demonstrations may be used to establish equations (29b) and (29c).

21. To Summarize:

If a system of charges moves in space in a known manner, the electric displacement $\mathbf{D}$ and the magnetic force $\mathbf{H}$ at any point in space may be derived from the retarded displacement potential Φ and the retarded vector potential $\mathbf{A}$ by the equations,

$$\mathbf{D} = -\operatorname{grad}\Phi - \frac{1}{s^2}\frac{d}{dt}\mathbf{A} \qquad (29)$$

$$\mathbf{H} = \operatorname{curl}\mathbf{A} \qquad (29)$$

The values of the retarded potentials Φ and $\mathbf{A}$ are to be computed by the equations,

$$\Phi = \frac{1}{4\pi}\int \frac{\left[\rho\right]_{(t-r/s)}}{r}\,dv \qquad (26)$$

$$\mathbf{A} = \frac{1}{4\pi}\int \frac{\left[\rho\mathbf{V}\right]_{(t-r/s)}}{r}\,dv \qquad (27)$$

IV. EXPRESSIONS FOR THE FORCES AT GREAT DISTANCES FROM THE RADIATOR

22. The Field at Points Near the Earth's Surface

We proceed to compute the electric displacement **D** and the magnetic force **H** at a point P near the earth's surface and at a great distance (100 wave lengths or more) from the radiator shown in Fig. 5. * In order that we may have specific conditions to discuss, let the radiator be assumed to be of the following proportions:

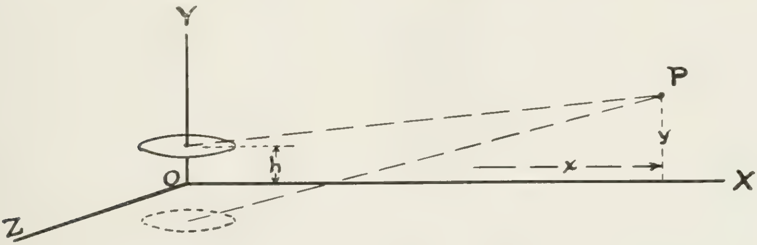

Fig. 12

Table I

Radius of charged areas	165 meters
Maximum elevation attained by the charged area	60 meters
Frequency	80,000 cycles
Distance to the point P	600 kilometers
Elevation of the point P above SS	90 meters
Maximum potential of + area to earth	100 kv.

The radiating system has been redrawn in Fig. 12. In this figure, the XZ plane has been taken to coincide with the neutral surface SS of Fig. 5; the XY plane has been passed through the point P = (x, y, O); and the positive and negative charges on the two sets of rings are assumed to be

*In the subsequent discussion, the following approximations are used: (a) the earth's surface is treated—not as a spherical surface—but as a plane surface, (b) the resistance losses in the surface of the earth and the absorption losses in the atmosphere have been neglected.

symmetrically distributed about the Y axis. The two sets of rings carry the charges $+Q$ and $-Q$ coulombs, and their respective elevations above the XZ plane are expressed by the equations:

$$h_p = h_o \cos \omega t \tag{30a}$$

$$h_n = -h_o \cos \omega t \tag{30b}$$

23. THE RETARDED DISPLACEMENT POTENTIAL Φ

As the element of charge (dQ) located at the center of the positive plate moves up and down in the simple harmonic manner expressed by equation (30a), its distance (r) from the point P varies in the manner expressed by the following equation.

$$r = \left[x^2 + (h_o \cos \omega t - y)^2\right]^{1/2}$$

$$= x\left[1 + \frac{(h_o \cos \omega t - y)^2}{2x^2} + \ldots\ldots\right]$$

Let us first draw the curves showing the steady state values of the displacement potential at P which correspond to the position of the charges at any instant of time, and then proceed to consider in what respect these curves must be altered to make them represent the retarded potential. By the "steady state value of the potential at P corresponding to the position of the charges at any instant" is meant the potential which would be assumed by the point P if the charges ever after remain fixed in the position they have at the instant under consideration.

The value at point P of the steady state displacement potential due to the various positions of the charge (dQ) is given by the equation,

$$d\,\Phi_p = \frac{1}{4\pi} \frac{dQ}{x\left[1 + \frac{(h_o \cos \omega t - y)^2}{2x^2}\right]}$$

$$= \frac{dQ}{4\pi x}\left[1 - \frac{(h_o \cos \omega t - y)^2}{2x^2}\right]$$

$$= \frac{dQ}{4\pi x}\left[1 - \frac{1}{2x^2}(h_o^2 \cos^2 \omega t - 2h_o y \cos \omega t + y^2)\right] \tag{31}$$

As the element of charge $(-dQ)$ located at the center of the negative plate moves up and down, its distance (r) from the point P varies in the manner expressed by the following equation,

$$r = \left[x^2 + (-h_o \cos \omega t - y)^2\right]^{1/2}$$

The value at the point P of the steady state displacement potential due to the various positions of this charge, is given by the equation,

$$d\,\Phi_n = -\frac{dQ}{4\pi x}\left[1 - \frac{1}{2x^2}(h_o{}^2 \cos^2 \omega t + 2h_o y \cos \omega t + y^2)\right] \quad (32)$$

The resultant displacement potential at P due to the corresponding elements of charge at the centers of the positive and negative sets of rings is

$$d\,\Phi = d\,\Phi_p + d\,\Phi_n = \frac{(dQ)\ h_o y}{2\pi x^3} \cos \omega t \quad (33)$$

Curves showing the variation in time of the values of the terms of equations (31), (32), and (33) have been drawn (but not to scale) in Figures 13, 14, and 15 respectively. For example, the right hand member of equation (31) consists of the constant term $\frac{dQ}{4\pi x}\left(1 - \frac{y^2}{2x^2}\right)$ upon which two very small variable terms are superimposed; the variable term $\frac{dQ}{4\pi x}\ \frac{h_o y \cos \omega t}{x^2}$ is of the same frequency as the frequency of vibration of the moving charge, and the term $-\frac{dQ}{4\pi x}$ $\frac{h^2{}_o \cos^2 \omega t}{2x^2}$ is of double this frequency. These terms are represented by curves A, B, and C of Fig. 13.

This question now arises: How must these curves be modified in order to make them represent the values of the retarded displacement potential at P?

At the instant at which the charges pass in opposite directions across the neutral surface (as the instant t_1), the distance of the elementary charges from P is $\sqrt{x^2 + y^2}$. Now the

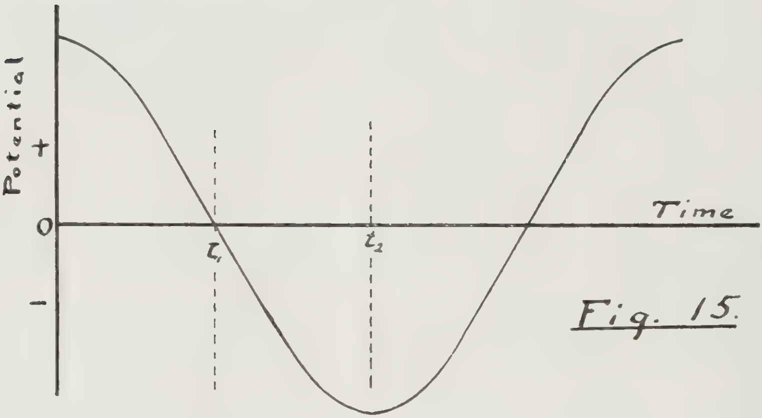

Fig. 15.

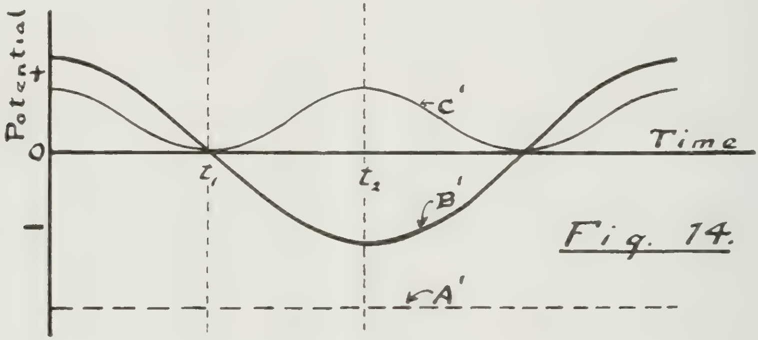

Fig. 14.

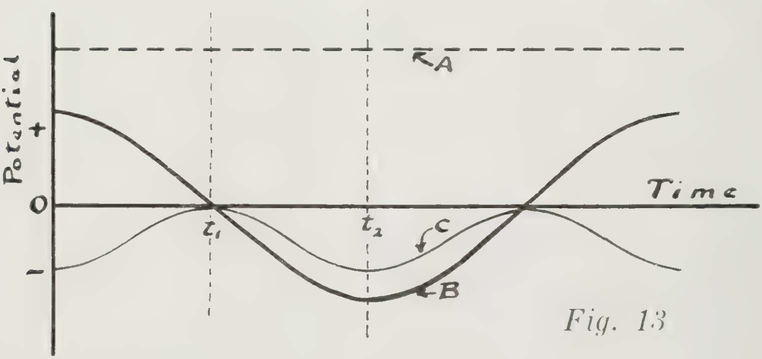

Fig. 13

effect corresponding to the location of any element of the charge at the instant (t_1) is felt at P (r/s) seconds later. Therefore it follows that the potential shown by Figures 13, 14, and 15 for the instant (t_1) represents the retarded potential at P at the instant $\left(t_1 + \frac{\sqrt{x^2+y^2}}{s}\right)$.

During the quarter cycle from (t_1) to (t_2), the positive charge moves farther and farther away from P and the negative charge approaches P. At the instant (t_2), the positive charge is at its maximum distance and the negative charge is at its minimum distance from P. At this instant the positive charge is approximately 1.2 cm. farther from, and the negative charge is approximately .6 cm. nearer to P than they were at the instant (t_1).

Therefore, the potential shown in Fig. 13 for the instant (t_2) represents the retarded potential at P at the instant $\left[t_2 + \frac{\sqrt{x^2+y^2}}{s} + \frac{1.2}{s}\right]$, while the potential shown in Fig. 14 for the instant (t_2) represents the retarded potential at P at the instant $\left[t_2 + \frac{\sqrt{x^2+y^2}}{s} - \frac{.6}{s}\right]$. That is to say, in altering the curves in Figs. 13, 14, and 15 in order to make them represent the retarded potentials, the potentials shown for the instant (t_1) in Figs. 13 and 14 must be shifted back in time by $\frac{\sqrt{x^2+y^2}}{s}$ seconds, the potential shown for the instant (t_2) in Fig. 13 must be shifted $\frac{\sqrt{x^2+y^2}}{s} + \frac{1.2}{s}$ seconds, and the potential shown for the instant (t_2) in Fig. 14 must be shifted $\frac{\sqrt{x^2+y^2}}{s} - \frac{.6}{s}$ seconds.

There is, therefore, between points on the Fig. 13 and Fig. 14 curves a relative shift which never exceeds $\left(\frac{1.2}{s} + \frac{.6}{s}\right)$ seconds, or $6 \; 10^{-11}$ seconds. Now a shift of $6 \; 10^{-11}$ seconds in time corresponds at a frequency of 80,000 cycles

to a shift of only five one-millionths of a cycle or .0018 degrees.

From this it follows that for all practical purposes,* the curves of Figures 13, 14, and 15 will represent the retarded potentials at P if they are shifted in time by $\frac{\sqrt{x^2+y^2}}{s}$ seconds, or substantially by $\frac{x}{s}$ seconds. From equation (33), it follows that the value of the retarded displacement potential at P due to the corresponding elements of charge at the centers of the positive and negative sets of rings is given by the expression:

$$d\,\Phi_r = \frac{(dQ)\,h_o y}{2\pi x^3}\cos\omega\left(t - \frac{x}{s}\right) \tag{34}$$

In like manner, every element of the positive charge may be paired with its corresponding element of the negative charge; any pair may be seen to contribute to the value of the retarded potential at P the quantity

$$d\,\Phi = \frac{(dQ)\;h_o y}{2\pi\;(x-X)^3}\cos\omega\left(t - \frac{(x-X)}{s}\right) \tag{35}$$

in which, X is the abscissa of the pair of charges under consideration. Thus, the resultant potential at P will be the sum of a great number of harmonic functions of the same frequency but differing somewhat in phase. The maximum difference in phase will be equal to the interval required for the transmission of the disturbance over a distance equal to the diameter of the radiator. This interval equals $330 \div 3\;10^8$ or $1.1\;10^{-6}$ seconds, which at a frequency of 80,000 cycles means a phase difference of .088 of a cycle, or 32 degrees.

Since the charges on the radiator have been assumed to be symmetrically disposed about the axis, and since the contribution made by any pair of elements to the potential

*It should be borne in mind that none of these arguments or conclusions apply if the point P is within a few wave lengths of the radiator, or if the line OP from the radiator to P makes an angle greater than one or two degrees with a horizontal plane. If OP makes an appreciable angle with the horizontal, these simple considerations require modification. See Appendix A for the forces at elevated points.

does not differ in phase by more than 16 degrees from the contribution of the central pair of charges, it follows that the value of the retarded displacement potential at P will, to within a few tenths of one per cent, be given by the expression:

$$\Phi = \frac{Q\, h_o\, y}{2\pi x^3} \cos \omega \left(t - \frac{x}{s}\right) \tag{36}$$

24. The Retarded Vector Potential $\mathbf{A}$

The retarded vector potential $\mathbf{A}$ is defined by the equation,

$$\mathbf{A} = \frac{1}{4\pi} \int \frac{\left[\rho \mathbf{V}\right]_{(t-r/s)}}{r} \, dv \tag{27}$$

Now, in Fig. 5, the charges move only in a direction parallel to the Y axis. Therefore, the X and Z components of $\mathbf{V}$ are zero, or the vector potential $\mathbf{A}$ is a vector parallel to the Y axis. For the positive charge, the Y component of the velocity $\mathbf{V}$ is,

$$V_{2p} = \frac{dh}{dt} = -h_o \omega \sin \omega t$$

and for the negative charge, the Y component of the velocity is,

$$V_{2n} = h_o \omega \sin \omega t$$

Since the moving charges are symmetrically disposed about the Y axis, and since the vector potential contributed by any element does not differ in phase by more than 16 degrees from the potential contributed by the charges moving along the Y axis, it follows that the value of the retarded vector potential at P will, to within a few tenths of one per cent, be given by the expression

$$\mathbf{A} = A_2 = \frac{Q}{4\pi x} \left\{ -h_o \omega \sin \omega \left(t - \frac{x}{s}\right) \right\} + \frac{-Q}{4\pi x} \left\{ h_o \omega \sin \omega \left(t - \frac{x}{s}\right) \right\}$$

$$\text{or } A_2 = -\frac{Q\, h_o \omega}{2\pi x} \sin \omega \left(t - \frac{x}{s}\right) \tag{37}$$

25. THE DISPLACEMENT AND POTENTIAL GRADIENT AT P.
From equation (28), the displacement is given by,

$$\mathbf{D} = -\operatorname{grad} \Phi - \frac{1}{s^2}\frac{d}{dt}\mathbf{A} \tag{28}$$

Taking the derivatives of Φ, as expressed in equation (36),

$$\frac{d\Phi}{dx} = \frac{Qh_0}{2\pi}\left[-\frac{3y}{x^4}\cos\omega\left(t-\frac{x}{s}\right)+\frac{y}{x^3}\frac{\omega}{s}\sin\omega\left(t-\frac{x}{s}\right)\right]$$

$$\frac{d\Phi}{dy} = \frac{Q\,h_0}{2\pi x^3}\cos\omega\left(t-\frac{x}{s}\right)$$

$$\frac{d\Phi}{dz} = 0$$

At points of the neutral surface (the XY plane), $\frac{d\Phi}{dx}$ is zero since (y) is zero. At points near the XY plane, $\frac{d\Phi}{dx}$ is negligibly small in comparison with $\frac{d\Phi}{dy}$. That is to say, the Y component of the gradient of Φ or D_2' is the only component of appreciable magnitude at points near the neutral surface. The Y component of the gradient of Φ is given by the expression,

$$D_2' = -\frac{d\Phi}{dy} = -\frac{Qh_0}{2\pi x^3}\cos\omega\left(t-\frac{x}{s}\right)$$

Since the X and Z components of the vector potential are zero, and by equation (37),

$$A_2 = -\frac{Q\,h_0\omega}{2\pi x}\sin\omega\left(t-\frac{x}{s}\right) \tag{37}$$

Therefore

$$D_1'' = D_3'' = 0$$

and

$$D_2'' \text{ or } -\frac{1}{s^2}\frac{d}{dt}A_2 = \frac{Q\,h_0\omega^2}{2\pi x s^2}\cos\omega\left(t-\frac{x}{s}\right)$$

Therefore the displacement at point P is parallel to the Y axis, and its value (vertically upward) is given by the expression,

$$D_2 = D_2' + D_2'' = \frac{Q h_o}{2\pi}\left[\frac{\omega^2}{s^2 x} - \frac{1}{x^3}\right]\cos\omega\left(t - \frac{x}{s}\right) \quad (38)$$

The potential gradient at P is likewise parallel to the Y axis and its value (vertically upward) is,

$$F_2 = \frac{D_2}{p} = \frac{Q h_o}{2\pi p}\left[\frac{\omega^2}{s^2 x} - \frac{1}{x^3}\right]\cos\omega\left(t - \frac{x}{s}\right) \quad (39)$$

26. The Magnetic Force at P

From equation (29), the magnetic force **H** is given by,

$$\mathbf{H} = \text{curl } \mathbf{A} \quad (29)$$

Now, at all points in space the X and Z components of the vector **A** are zero, and at the point P, the value of **A** is,

$$\mathbf{A} = A_2 = -\frac{Q h_o \omega}{2\pi x}\sin\omega\left(t - \frac{x}{s}\right) \quad (37)$$

Therefore,

$$H_1 \text{ or } \frac{dA_3}{dy} - \frac{dA_2}{dz} = 0$$

$$H_2 \text{ or } \frac{dA_1}{dz} - \frac{dA_3}{dx} = 0$$

$$H_3 \text{ or } \frac{dA_2}{dx} - \frac{dA_1}{dy} \text{ is,}$$

$$H_3 = \frac{Q h_o}{2\pi}\left[\frac{\omega^2}{x s}\cos\omega\left(t - \frac{x}{s}\right) - \frac{\omega}{x^2}\sin\omega\left(t - \frac{x}{s}\right)\right] \quad (40)$$

That is to say, the magnetic force is exerted in a direction which is normal to the plane determined by the point P and the axis of the radiator. The magnetic force is exerted along circular paths which are centered upon the axis of the radiator, and which lie in planes normal to this axis.

The magnetic flux density at P will be given by the expression:

$$B_3 = \frac{Q h_o \mu}{2\pi}\left[\frac{\omega^2}{x s}\cos\omega\left(t - \frac{x}{s}\right) - \frac{\omega}{x^2}\sin\omega\left(t - \frac{x}{s}\right)\right] \quad (41)$$

V. APPLICATION OF THE EXPRESSIONS FOR THE FORCES AT DISTANT POINTS

27. The Electric Force

The expression for the electric force at P—equation (39)—contains two terms. As the distance from the radiator to the point P is increased, the first term decreases as the first power of the distance and the second term decreases as the cube of the distance. At zero frequency, that is, with the charges stationary, the first term disappears from the equation, and the second term is seen to be identical with the expression previously derived for the gradient in the electrostatic field by the method of images. Hence, we may call the first term the "radiant term" and the second term the "steady-state term."

The radiant and steady-state terms are of equal magnitude at the point $x = \frac{s}{\omega}$, or at a point .159 of a wave length from the radiator. (It should be noted, however, that equations (39) and (40) do not apply to points this close to the radiator). For all points at a greater distance than this from the radiator, the radiant term is the larger term; at great distances, the steady-state term becomes negligibly small. For example: at a point P which is at a distance of 600 kilometers from the radiator specified in Table I, the electric force or potential gradient is

$$F_2 \text{ (in volts per cm.)} = \left[\frac{6.2}{10^5} - \frac{6.2}{10^{11}}\right] \cos 500{,}000\,(t - .002)$$

The magnitude of the radiant term at this point is seen to be one million times as great as the steady-state term.

For points at a great distance from a radiator vibrating at the usual wireless frequencies, the steady-state term may be dropped and equation (39) may be written thus:

$$F_2 = \frac{240\pi^2}{s} \frac{Q\ h_o\ f^2}{x} \cos \omega \left(t - \frac{x}{s}\right) \qquad (42)$$

28. The Magnetic Force

The expression for the magnetic force at P equation (40)—also consists of two terms. As the distance from the radiator to the point P is increased, the first term decreases as the first power of the distance, and the second term decreases as the square of the distance. Both terms vanish for zero frequency. The coefficient of the second term, namely $\frac{Q\ h_o\ \omega}{2\pi x^2}$ is the magnetic force which would be set up at P if the two charges, instead of oscillating, were to move continuously away from each other with a uniform velocity equal to the velocity they have when crossing the neutral surface. Therefore, we may call the first term the "radiant" term and the second the "steady-current" term.

As in the case of the electric force, the two terms in the magnetic force are of equal magnitude at a point .159 of a wave length from the radiator. At greater distances the radiant term is the larger term. For example: At a point P which is 600 kilometers from the radiator specified in Table I, the magnetic force is

$$H_3 \text{ (in ampere-turns per cm.)} = \frac{1.66}{10^7} \cos \omega \left(t - \frac{x}{s}\right) + \frac{1.66}{10^{10}} \sin \omega \left(t - \frac{x}{s}\right)$$

The magnitude of the radiant term at this point is one thousand times as great as the "steady-current" term. For points at great distances from a radiator vibrating at the usual wireless frequencies the steady current term may be dropped and equation (40) may be written,

$$H_3 = \frac{2\pi}{s} \frac{Q\ h_o\ f^2}{x} \cos \omega \left(t - \frac{x}{s}\right) \tag{43}$$

29. The Radiant Vector

By Poynting's Theorem, the rate $\mathbf{P}_1$ at which energy streams across unit area at P is the vector product of the electric and magnetic forces at P.

$$\mathbf{P}_1 = \mathbf{F} \times \mathbf{H} \text{ (watts per sq. cm.)}$$

Using the approximate values for **F** and **H** given in equations (42) and (43) and taking the vector product, it is seen that the radiant vector $\mathbf{P}_1$ at P points radially away from the oscillator and has the value,

$$P_1 = \frac{480\pi^3}{s^2} \frac{Q^2 h_o^2 f^4}{x^2} \cos^2 \omega \left(t - \frac{x}{s}\right) \tag{44}$$

For the specific conditions of Table I, this reduces to

$$P_1 \text{ (in watts per sq. cm.)} = \frac{1.01}{10^{11}} \cos^2 \omega \left(t - \frac{x}{s}\right)$$

This result may be arrived at in another way. The electro-potential energy per unit volume at P is

$$\frac{1}{2} pF^2 \text{ or } \frac{240\,\pi^3}{s^3} \frac{Q^2 h_o^2 f^4}{x^2} \cos^2 \omega \left(t - \frac{x}{s}\right)$$

The electro-kinetic energy per unit volume at P is

$$\frac{1}{2} \mu H^2 \text{ or } \frac{240\pi^3}{s^3} \frac{Q^2 h_o^2 f^4}{x^2} \cos^2 \omega \left(t - \frac{x}{s}\right)$$

The energy in the electro-potential form per unit volume is seen to equal the energy in the electro-kinetic form, and the total energy per unit volume is $\frac{480\pi^3}{s^3} \frac{Q^2 h_o^2 f^4}{x^2} \cos^2 \omega \left(t - \frac{x}{s}\right)$ Since the state of the medium is propagated outward with the velocity (s), the rate at which energy streams across unit area at P will be (s) times the energy per unit volume.

$$s\left[\frac{480\pi^3}{s^3} \frac{Q^2 h_o^2 f^4}{x^2} \cos^2 \omega \left(t - \frac{x}{s}\right)\right] = \frac{480\pi^3}{s^2} \frac{Q^2 h_o^2 f^4}{x^2} \cos^2 \omega \left(t - \frac{x}{s}\right)$$

This product is seen to be identical with the expression in equation (44).

30. The "Radiation Figure of Merit" of an Antenna

We are now in a position to answer the question raised at the beginning of this discussion, namely: How is the magnitude of the disturbance which is set up in the medium at a

distant point P affected by the height of the capacity area of the radiating station?

The expressions for the electric force, for the magnetic force, and for the rate at which energy streams past P—equations (38) to (41)—all contain the factor (Qh_0). That is, the magnitudes of the forces and fluxes are directly proportional to the value of the product of the charge (Q) on the capacity area times the maximum height (h_0) of the capacity area, and the rate at which energy streams past P is proportional to the square of this product.

Now the charge Q on any capacity area may be written as equal to the capacity (C) of the area to earth times the voltage (E) from the charged area to earth, both C and E being the values for the instant when the area is at its point of maximum elevation. Accordingly, (Qh_0) may be replaced by its equivalent (Ch_0E). Now the maximum voltage (E) which may be applied between the capacity area and earth is limited by such considerations as the failure of the insulators and the ionization losses around the wires. This voltage is (within limits) substantially independent of the capacity (C) and elevation (h_0). Possibly the limiting voltage would be somewhat lower for a radiator having the capacity area at a great elevation (150 meters) than for a radiator having the capacity area at a moderate elevation (10 meters), because of the greater mechanical difficulties encountered in insulating an extended area at a great elevation.

From this it follows that if two different antennae of extended area are operated at the same voltage and frequency, then the magnitudes of the electric forces—or of the magnetic forces—at the same distance from the two antennae are proportional to the values of the (Ch) products of the two antennae. The product Ch—the product of the capacity of the extended area times its elevation—may, therefore, be called the *Radiation Figure of Merit* of the antenna.

The question now is, How is the figure of merit of an antenna having an extended capacity area affected by the height of the capacity area above the ground? Imagine the area to be in the form of a circular sheet of radius R, as in Fig. 1. If the radius (R) is large as compared with the eleva-

tion (h), the capacity (C) is given approximately by the expression

$$C = \frac{p\pi R^2}{h}$$

$$\text{Whence } Ch = p\pi R^2$$

That is to say, the radiation figure of merit of the antenna is independent of h, or the values of the electric and magnetic forces at distant points are independent of the height of the capacity area. On the other hand, if the radius R is small in comparison with h, the capacity is given approximately by the expression,

$$C = 8pR$$

$$\text{Whence } Ch = 8pRh$$

That is, the radiation figure of merit of the antenna is directly proportional to the elevation h. Between these extreme conditions, the figure of merit of an antenna of given area increases with the elevation of the capacity area, but at a lower rate than the first power of the elevation.

This leads us to a brief consideration of the feasibility of mounting the capacity areas in wireless telegraph stations at a low elevation.

31. A Comparison of Specific Examples of High and Low Antennae

As typical examples of low and high antenna, let the radiator specified in Table I (mounted, however, at only 10 meters elevation) be compared with the antenna of the Government Wireless Station at Arlington, D. C.*

As previously stated, the capacity area of the Arlington Station consists of three approximately horizontal wire harps suspended at an elevation of approximately 150 meters. The capacity of this antenna as determined by measurement is reported to be .0094 microfarads. The antenna cannot, however, be regarded as the equivalent of a radiating area at an elevation of 150 meters and having a capacity of .0094 microfarads. To find the real capacity of an

*For a description of the Arlington Station see D. W. Todd, *Jour. Am. Soc. Naval Engineers*, Vol. 25, February 1913. Also Kintner, Forbes, Kroger and Hogan, *United States Navy Wireless Station* in *Electrical World*, Vol. 61, page 721, April 1913.

area which, mounted at an elevation of 150 meters, would be the radiating equivalent of the Arlington antenna, one must subtract from the measured value of the Arlington antenna to allow for such effects as the following:

1st. A part of the capacity of .0091 microfarads is through the insulators to the steel towers. Any displacement current through the insulators is accompanied by a conduction current flowing up and down the steel supporting towers. Again, a portion of the electrostatic flux from the wire harps terminates on the upper portions of the steel towers; this leads to conduction currents in the towers. These charges running up and down the steel towers move in the opposite direction to the charges in the tail of the antenna and so partially neutralize the radiation from the antenna.

2d. The .0091 microfarads includes the capacity due to displacement from the lower portions of the antenna tail to ground. This capacity area is of course not very effective because of its low elevation.

I should estimate that the radiating power of the Arlington antenna is not greater than that of a capacity area at an elevation of 150 meters and having a capacity of .005 microfarads. The radiation figure of merit of the Arlington antenna is therefore estimated to be .005 x 150 = .75 microfarad-meters. The capacity area specified in Table I is a circular sheet of 165 meters radius mounted at an elevation of 60 meters. The capacity of this sheet to earth is approximately .0125 microfarads, and its figure of merit is .75 microfarad-meters. The radiator of Table I if operated at the same voltage and frequency as the Arlington antenna may, therefore, be expected to set up at distant points a field of the same intensity as the Arlington antenna. Moreover, this sheet of 165 meters radius if mounted at a height of 5 or 10 meters will set up the same field as if mounted at the height of 60 meters.

By mounting the capacity area at an elevation of 10 meters, its capacity is increased to .075 microfarads. This is only 40 per cent less than the capacity of the compressed air condensers used in the primary oscillation circuit of the Arlington station. Now the antenna can very readily be insulated for operating voltages far in excess of the voltages

usually applied to the condensers in the primary circuit (30 to 50 Kv). Therefore, if the capacity area is mounted at the moderate elevation of 5 or 10 meters, it is possible to store in the antenna circuit more energy than is generally stored in the condensers of the primary circuits of high power wireless stations. This makes it feasible to dispense with the coupled circuits which are at present used in spark systems of wireless telegraphy and to obtain slightly damped oscillations from a simple oscillating circuit comprising an extended capacity area M, an inductance L, and a spark gap S as shown in Fig. 16.

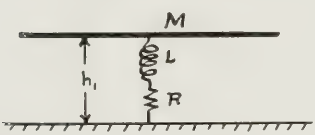

FIG. 17

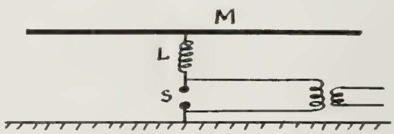

FIG. 16

32. CONSTANTS OF A SENDING STATION

The constants of an antenna of the dimensions given in Table I, save that the capacity area is mounted at an elevation of only ten meters, would be as follows:

TABLE II

CONSTANTS OF AN ANTENNA 10 METERS IN HEIGHT

Radius of capacity area	165	meters
Height of capacity area	10	meters
Capacity of area to earth	.075	microfarads
Assumed operating conditions		
Voltage at moment of discharge	100	peak Kv
Frequency of oscillation	80,000	cycles per sec.
For the above voltage & frequency		
Energy stored	375	joules
Power input at 1000 sparks per second	375	Kw.
Peak value of antenna current	3800	amperes
Critical resistance of oscillatory circuit	53	ohms
Initial rate of radiation	80.	Kw
Radiation resistance (See Appendix B)	.011	ohms
Logarithmic decrement per cycle due to radiation of energy	.0013	
Linear decrement per cycle due to radiation of energy	.13	per cent

It will be noted that the decrement due to the radiation of energy is extremely low: the decrease in the voltage or current due to radiation losses is only .13 of one per cent per cycle. In addition to the radiation loss, the following losses require consideration: the I^2R loss in the conductors, the ionization losses in the air around the conductors, and the losses in the spark gap. The resistance of the conductors to the 80,000 cycle currents can readily be made less than .01 ohms, and the ionization losses made negligibly small. The equivalent resistance of the spark will be of the order of .1 ohms (between .05 and .15 ohms). If the equivalent resistance of the spark is estimated to be .1 of an ohm, the total damping resistance is .12 ohms. This resistance will cause a decrement of only 1.4 per cent per cycle.

The current and the rate of radiation in Table II are calculated for a peak voltage of 100 Kv. This is considerably in excess of the voltages which have hitherto been used with rotary spark gaps. There may be some uncertainty as to the performance of a rotary spark gap at this voltage.

VI. THE LOW ANTENNA FOR RECEIVING PURPOSES

33. The Induced Voltage

Thus far the low antenna has been discussed only as a radiator of electromagnetic waves. This question now arises: Is the effectiveness of an extended antenna as an absorber or receiver of electromagnetic energy independent (within the limits previously stated) of its height above the ground?

Imagine the receiving station to comprise an extended capacity area M, (Fig. 17) mounted at the height (h_1) above the ground, an inductance L, and receiving devices of ohmic resistance R. (The receiving devices may be inductively coupled with the simple series circuit shown in Fig. 17; in this case, R represents the resistance of the receiving devices reduced to the primary circuit). Assume the circuit shown in Fig. 17 to be resonant to the frequency of the sending station, and to be located at the distance (x) from the sending station.

From equation (42), the potential gradient at the receiving station is

$$F_2 = \frac{240\pi^2}{s}\frac{Q\,h_o\,f^2}{x} \cos\omega\left(t - \frac{x}{s}\right) \tag{42}$$

Therefore the voltage E_1 between the plate M of the receiving station and earth is

$$E_1 = \frac{240\pi^2}{s}\frac{Q\,f^2\,h_o h_1}{x} \cos\omega\left(t - \frac{x}{s}\right) \tag{45}$$

34. THE BUILDING UP OF THE OSCILLATION

Neglecting minor terms, the equation for the start of an alternating current in a highly oscillatory circuit *resonant* to the impressed frequency, and comprising resistance, inductance, and capacity in series is as follows:

Measuring time from the instant at which the electromotive force is impressed in the circuit, and representing the impressed electromotive force (e) by the equation

$$e = E\cos\omega\,(t - t_1)$$

the equation for the current (i) in the circuit is

$$i = \frac{E}{R}\left(1 - \varepsilon^{-\frac{Rt}{2L}}\right)\cos\omega\,(t - t_1) \tag{46}$$

in which

R is the resistance of the resonant circuit, and
L is the inductance of the resonant circuit.

Now the resistance of the receiving station consists of—
(1) the useful resistance R_1, that is, the equivalent resistance of the devices which consume a part of the received energy and are actuated thereby.
(2) the unavoidable ohmic resistance R_2 of the antenna conductors and ground connections.
(3) the radiation resistance R_3 of the antenna.

Item (2) can be made smaller than Item (3), and of two stations operating at the same frequency, the station having the higher radiation resistance would, in general, have the higher Item (2) resistance. That is, the two items (2) and (3) may be regarded as very roughly proportional. Let us, therefore, lump items (2) and (3), and denote their sum by

$R_0 = R_2 + R_3$. R_0 will consist mainly of the radiation resistance R_3, the expression for which is,

$$R_3 = \frac{160\pi^2 h_1^2 f^2}{s^2} \text{ ohms*} \tag{47}$$

From this expression it will be noted that the radiation resistance of the antenna is proportional to the square of the height of the capacity area.

Let us first compare the ultimate values to which the current and voltage build up, and the rate at which they build up to these values, in two receiving stations in which the only resistance is the resistance R_0. Let station A have the capacity area mounted at the height (h_1) and station B at the height (nh_1), and let the antennae be of the same area in the two stations. The constants of the two circuits are as shown in Table III.

TABLE III

Relative Constants of Low and High Antennae

	Station A	Station B
Height of capacity area	h_1	nh_1
Induced voltage (peak)	E	nE
Capacity	C	$\frac{C}{n}$
Inductance	L	nL
Radiation resistance	R	n^2R
Final current (peak)	$\frac{E}{R}$	$\frac{E}{nR}$
Final condenser voltage (peak)	$\frac{E}{RC\omega}$	$\frac{E}{RC\omega}$
Final energy stored $\frac{1}{2}LI^2$	$\frac{LE^2}{2R^2}$	$\frac{LE^2}{2nR^2}$
or $\frac{1}{2}CE^2$	$\frac{E^2}{2R^2C\omega^2}$	$\frac{E^2}{2R^2nC\omega^2}$
Final rate of radiation	$\frac{E^2}{R}$	$\frac{E^2}{R}$
Time constant	$\frac{2L}{R}$	$\frac{2L}{nR}$

*See Appendix B.

It is seen that the receiving properties of the two antennae correspond with their relative radiating properties. The current in the low antenna builds up to (n) times the current in the high: both build up to the same condenser voltage: therefore, the low antenna stores (n) times as much energy as the high, and has a time constant (n) times as long as that of the high antenna. The final rate of radiation is the same for both stations. That is, both antennae ultimately abstract energy at the same rate from the passing electromagnetic waves, but the high antenna abstracts energy at a greater rate than the low antenna during the initial stages (first few swings) of the oscillation. The high antenna will, therefore, respond much more readily to highly damped waves than will the low antenna. This means, of course, that when receiving undamped or slightly damped waves, the high antenna will be subject to greater interference from atmospheric disturbances than will the low antenna. To reduce this interference in the case of stations with high antenna, additional capacity is used in the "interference preventer" circuits. In other words, the low antenna is to be regarded as the equivalent of a high antenna and "interference preventer" combined.

Let us now suppose that the two stations under comparison contain receiving devices. The maximum amount of power is expended in these devices if the resistance R_1 of the utilizing devices is made equal to the resistance R_0. With the resistence R_1 so proportioned, the final rate at which energy is abstracted from the passing waves by the antenna is half as great as it would be if the utilizing devices were left out of the circuit. Of the energy so abstracted, one half is expended in the utilizing devices and one half is re-radiated. That is, in the case of both the high and the low antenna, the maximum rate at which energy can be expended in the utilizing devices is approximately equal to one quarter of the final rate of radiation with the utilizing devices cut out of the circuit. Since—as shown in Table III—this rate of radiation is the same for stations A & B, the conclusions previously drawn as to the relative receiving properties of the two stations are not altered by the insertion of utilizing devices with properly proportioned resistances.

35. Computed Value of the Received Power

If the station whose constants are given in Table II is used to emit undamped waves and if a similar station is used for receiving, then the maximum amount of power will be expended in the utilizing devices if their equivalent resistance is approximately .02 ohms. With such a resistance and with a distance of 5000 kilometers between the two stations and no absorption losses in transmission, the computed final values of the power received, current, etc. in the receiving station are given in Table IV.

Table IV

Computed value of received power with similar sending and receiving stations

Height of capacity areas	10	meters
Capacity of antenna to earth	.075	microfarads
Distance between stations	5000.	Km
Assumed sending conditions		
Voltage	100	peak Kv
Current	3800	peak amperes
Frequency	80000	cycles per sec.
Rate of radiation	80	Kw
Receiving Station		
Radiation plus wire & earth resistance	.02	ohms
Resistance of utilizing devices	.02	ohms
Induced voltage	.0076	peak volts
Final condenser voltage	5.0	peak volts
Final current	.19	peak amperes
Final expenditure in utilizing devices	.00035	watts
Inductance	53.	microhenries
Time constant	.0026	seconds
Time constant	210.	cycles

VII. SUMMARY

36. To Summarize:

1st. If an electro-magnetic radiator having a capacity area with a radius which is large in comparison with any feasible mounting elevation is operated at a given voltage and frequency, its radiation figure of merit is independent of the elevation at which the capacity area is mounted.

2nd. It is feasible to construct a radiator with the capacity area at a very moderate elevation which will have a radiation figure of merit equal to or greater than the values which are at present attained in long distance wireless stations by mounting the capacity area at a great elevation.

3rd. An elevated antenna will respond more readily to rapidly damped oscillations than will a low antenna. Both antennae (if the capacity areas are equal) will ultimately absorb the same amount of power from sustained or very slightly damped trains of waves. Therefore, when receiving sustained oscillations, the low antenna may be less subject to interference from atmospheric disturbances and other stations.

4th. The relative advantages of low versus high antennae for high power radio telegraph stations have been tabulated below.

Advantages or Merits of Low Versus High Antennae for High Power Stations

Low Antenna	High Antenna
Lower first cost (except where the cost of land per acre is very high) Power condensers are unnecessary Single frequency of oscillation. Apparently possible to obtain a smaller decrement than where power condensers are necessary. Less likelihood of damage by lightning. Probably less interference from "atmospheric."	Smaller antenna current—this may be of considerable advantage where arc generators or high frequency alternators are used A smaller number of insulators will be required: less likelihood of interruption due to insulator failures.

APPENDIX A

THE FORCES AT POINTS AT A GREAT DISTANCE FROM THE RADIATOR AND AT ANY ELEVATION ABOVE THE NEUTRAL PLANE

37. Let P in Fig. 18 represent any point at a great distance from the radiator, and let Q represent any small portion of the positive moving charge.

We wish to calculate the potentials and the forces at the point at the instant t which arise from this moving charge and its image −Q.

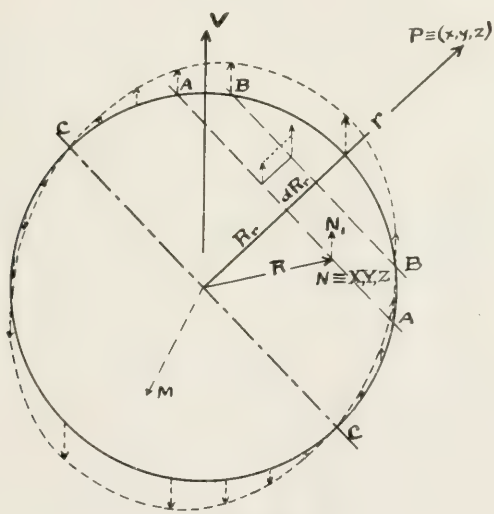

Fig. 18

Let some element in the moving charge Q be selected, and let us suppose that this element reaches the point O in space at such an instant (t_o) previous to (t) that the distance OP will, at the velocity of propogation (s) in the intervening medium, be traversed in the interval $(t - t_o)$. Then

$$OP \text{ or } r = s\,(t - t_o) \qquad (48)$$

Let O (regarded as a point fixed in space) be chosen as the origin of coordinates and let the coordinates of P be represented by x, y, z.

Let us take any other element of the charge, as the element N, whose coordinates at the instant (t_o) are X, Y, Z, and determine its coordinates when it is occupying such a position that the influence propogated from the element N will reach the point P at the instant (t), that is, simultaneously with the influence from the element at O. This position will be called "the effective position of the element N corresponding to the instant (t)". Let this effective position be represented by N_1 $(= X_1, Y_1, Z_1.)$

Assume that this effective position N_1 is reached at an instant (t_1) which is later than the instant (t_o) by the interval τ.

Then

$$t_1 = t_o + \tau \tag{49}$$

The condition that N_1 shall be the effective position of the element is that the distance

$$N_1P \text{ shall equal } s(t-t_1) = s(t-t_o-\tau) \tag{50}$$

If the charge Q is assumed to be all moving in the same direction with the velocity **V** (components V_1, V_2, and V_3), then

$$\left.\begin{aligned} X_1 &= X + V_1\tau \\ Y_1 &= Y + V_2\tau \\ Z_1 &= Z + V_3\tau \end{aligned}\right\} \tag{51}$$

But

$$\begin{aligned} N_1P &= \sqrt{(x-X_1)^2+(y-Y_1)^2+(z-Z_1)^2} \\ &= \sqrt{(x-X-V_1\tau)^2+(y-Y-V_2\tau)^2+(z-Z-V_3\tau)^2} \end{aligned} \tag{52}$$

Therefore, from (50) and (52),

$$s^2(t-t_o-\tau)^2 = (x-X-V_1\tau)^2+(y-Y-V_2\tau)^2+ (z-Z-V_3\tau)^2 \tag{53}$$

Expanding, and dropping infinitesimals of higher orders,

$$\tau\left(s-\frac{xV_1+yV_2+zV_3}{r}\right) = \frac{xX+yY+zZ}{r}$$

But $\frac{xV_1 + yV_2 + zV_3}{r}$ is the component of the velocity of the charge in the direction OP, or along the radius (r) to the point P.

Representing $\frac{xV_1 + yV_2 + zV_3}{r}$ by V_r. (54)

$$\tau = \frac{1}{s - V_r} \frac{xX + yY + zZ}{r} \qquad (55)$$

It will be noted that $\frac{xX + yY + zZ}{r}$ is the projection of the radius ON on the radius OP.

Representing $\frac{xX + yY + zZ}{r}$ by R_r, (56)

equation (55) may be written

$$\tau = \frac{R_r}{s - V_r} \qquad (57)$$

The element which at the instant (t_0) has the coordinates N_1 (= X, Y, Z.) has its effective position corresponding to the instant (t) at the point $N_1 = X_1, Y_1, Z_1$, and its distance from P, namely N_1P is (by equations (50) and (57)).

$$N_1P = s\,(t - t_0 - \tau) = r - s\,\tau = r - \frac{sR_r}{s - V_r} \qquad (58)$$

38. Let us now assume that the moving charge Q is spherical, that the element O is the center of the charge, and that the integrations indicated in equations (26) and (27) for the retarded potentials Φ and $\mathbf{A}$ are to be carried out over this charge for the instant (t). It will be noted that the volume occupied by the charge in its "effective position corresponding to the instant (t)" (that is, the volume over which the integrations are to be taken) is not spherical in shape. This is due to the fact that the influence of any element of the charge, as N, which is nearer to the point P than is the center O of the sphere, reaches P before the influence from O. The effective position of N is therefore some position, as N_1,

which it occupies later in time. That is, the effective position of any element whose distance from P is less than OP will be found by displacing the element in the direction of movement of the charge by an amount which in the case of slowly moving charges will be shown to be proportional to the distance R_r of the element from the plane CC, and to the velocity of the moving charge. On the other hand, the effective position of any element M whose distance from P is greater than OP will be found by displacing the element in a direction opposite to the direction of movement of the charge. The integrations indicated in equations (26) and (27) for the retarded potentials must, therefore, be carried on over the space found by distorting the sphere as indicated by the dotted outlines of Fig. 18.

The relative volume of the spherical space and the space within the dotted outline may be found as follows. Consider a differential slice of the charge included between two planes AA and BB perpendicular to the radius OP. Let the projection on OP of the radius to any point in the plane AA be R_r, and the projection of any point in BB be $R_r + dR_r$. The effective position of any element in the AA plane is given by equations (51) and (57) as

$$X_1 = X + V_1 \tau = X + \frac{R_r}{s - V_r} V_1$$

$$Y_1 = Y + V_2 \tau = Y + \frac{R_r}{s - V_r} V_2$$

$$Z_1 = Z + V_3 \tau = Z + \frac{R_r}{s - V_r} V_2$$

That is, in their effective positions corresponding to the instant (t) all elements in the AA plane are displaced in the direction of movement of the charge by the amount $\frac{V}{s - V_r} R_r$ In like manner, all elements in the BB plane are displaced by the amount $\frac{V}{s - V_r} (R_r + dR_r)$

The displacement of the BB plane is therefore greater than that of the AA plane by the amount $\frac{V}{s - V_r} dR_r$. Since

this difference is between displacements in the direction of motion of the charge, the *increase* in the *perpendicular* distance between the planes is $\frac{V_r}{s-V_r} dR_r$

Hence the volume between the planes in their effective positions is equal to $\left(1+\frac{V_r}{s-V_r}\right) = \left(1+\frac{V_r}{s} - \frac{V_r^2}{s^2} \ldots\ldots\right)$ times the volume between the planes AA and BB. That is, the effective volume over which the integration is to be taken is greater than the spherical volume in the ratio of $\left(1+\frac{V_r}{s} - \frac{V_r^2}{s^2} \ldots\ldots\right)$ to 1.

39. For the case of slowly moving charges, as in the low frequency station whose constants are discussed in *Section 31*, we may now repeat the arguments used in *Sections 22 to 21*, save that the charge must be multiplied by $\left(1+\frac{V_r}{s} - \frac{V_r^2}{s^2} \ldots\right)$ and that P must be taken to represent a point at any elevation above the neutral plane and at a great distance from the radiator.

Letting θ represent the angle between the line OP and the axis of the radiator, the following expressions are obtained for the retarded potentials at P.

$$\Phi = \frac{Qh_o}{2\pi r}\left[-\frac{\omega \cos\theta}{s} \sin\omega\left(t-\frac{r}{s}\right) + \frac{\cos\theta}{r}\cos\omega\left(t-\frac{r}{s}\right)\right] \tag{59}$$

$$\mathbf{A} = A_2 = -\frac{Q h_o f}{r}\sin\omega\left(t-\frac{r}{s}\right) \tag{60}$$

From these expressions have been omitted the terms which in the case of slowly moving charges, are negligible at great distances from the radiator.

From these expressions for the retarded potentials, the following expressions may be derived for the *radiant* terms in the electric and magnetic force at P.

$$H_1 = H_2 = O$$

$$H_3 = \frac{2\pi Q h_o f^2}{sr} \sin\theta \cos\omega\left(t - \frac{r}{s}\right) \tag{61}$$

$$F_1 = -\frac{240\pi^2 Q h_o f^2}{sr} \cos\theta \sin\theta \cos\omega\left(t - \frac{r}{s}\right) \tag{62}$$

$$F_2 = \frac{240\pi^2 Q h_o f^2}{sr} \sin^2\theta \cos\omega\left(t - \frac{r}{s}\right) \tag{63}$$

$$F_3 = 0$$

The resultant electric force (the resultant of F_1 and F_2) is seen to be normal to the radius OP, to have the absolute value F given below, and to lie in the plane determined by the point P and the axis of the radiator.

$$F = \frac{240\pi^2 Q h_o f^2}{sr} \sin\theta \cos\omega\left(t - \frac{r}{s}\right) \tag{64}$$

Since the magnetic force is perpendicular to this plane, the electric and magnetic forces are both perpendicular to the line OP, and they are perpendicular to each other. Therefore, the radiant vector is in the direction of the line OP, and the rate P_1 at which energy streams out across a plane perpendicular to OP at the point P is

$$P_1 = \frac{480\pi^3}{s^2} \frac{Q^2 h_o^2 f^4}{r^2} \sin^2\theta \cos^2\omega\left(t - \frac{r}{s}\right) \text{ (watts per sq. cm.)} \tag{65}$$

APPENDIX B

THE RATE OF RADIATION AND THE RADIATION RESISTANCE

40. Rate of Radiation

At any point at a great distance from the radiator, the direction of flow of energy is normal to the surface of a sphere passing through the point and having the radiator as a center. The rate P_1 at which energy passes outward across a square centimeter of the surface of the sphere at the point is given in equation (65). The rate of radiation is a maximum in the equatorial plane of the radiator and falls off toward the pole as the square of the sine of the angular distance of the point from the pole.

The *mean* rate of radiation P from the radiator (with undamped oscillations) will be found by integrating over the entire surface of a hemisphere described about the radiator, and then writing the mean value for a cycle of the expression so obtained. Upon carrying out this integration, the following expression is obtained for the rate of radiation P through a hemisphere described about the radiator.

$$P = \frac{320\pi^4 Q^2 h_o^2 f^4}{s^2} \text{ watts} \tag{66}$$

41. Radiation Resistance

The charges depicted in Fig. 5 *each* cross the neutral plane 2f times per second. The quantity of electricity which crosses the neutral plane per second is, therefore, 4fQ coulombs, or the moving charges convey the same quantity as an alternating current whose average value is 4fQ amperes, and whose r. m. s. value (I) is

$$I = \sqrt{2}\ \pi f Q \quad \text{r. m. s. amperes} \tag{67}$$

Now the "radiation resistance" may be defined as a fictitious resistance of such a value that the product of the radia-

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tion resistance times the square of the current will equal the rate of radiation from the radiator. That is, the radiation resistance R_3 is defined by the equation

$$R_3 = \frac{P}{I^2}$$

Substituting the values of P and I as given in equations (66) and (67),

$$R_3 = \frac{160\,\pi^2 f^2 h_0^2}{s^2} \tag{68}$$

42. Logarithmic Decrement Due to the Radiation of Energy

The logarithmic decrement (δ) of the oscillation is defined as the Naperian logarithm of the ratio of any peak value to that following it by a cycle.

Let it be assumed that the oscillation of the radiator is not sustained but is damped, and that the only loss is that due to radiation. For a slightly damped circuit containing resistance, inductance and capacity in series, the value of the logarithmic decrement is given quite accurately by the expression

$$\delta = \frac{2\pi\ R}{R_c} \tag{69}$$

in which,

R is the actual resistance of the circuit, and

R_c is its critical resistance, defined by the equation

$$R_c = 2\sqrt{\frac{L}{C}} = \frac{2}{2\pi fC}$$

where f is the natural frequency of oscillation of the circuit.

Now, in the case of a low extended antenna of area A, the capacity C is given approximately by the expression.

$$C = \frac{pA}{h_o}$$

Whence

$$R_c = \frac{120\ s\ h_o}{fA} \tag{70}$$

Substituting in equation (69) the value of R_c as given in equation (70) and for R the value of R_3 as given in equation (68), the following expression is obtained for the logarithmic decrement.

$$\delta = \frac{8\pi^3\ f^3\ h_o A}{3\ s^3} \tag{71}$$

BULLETIN OF THE UNIVERSITY OF WISCONSIN

NO. 879

ENGINEERING SERIES VOL. 8. NO. 5, PP. 247-332

THE PHYSICAL PROPERTIES OF MAGNESIA CEMENT AND MAGNESIA CEMENT COMPOUNDS

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MADISON, WISCONSIN
NOVEMBER, 1917

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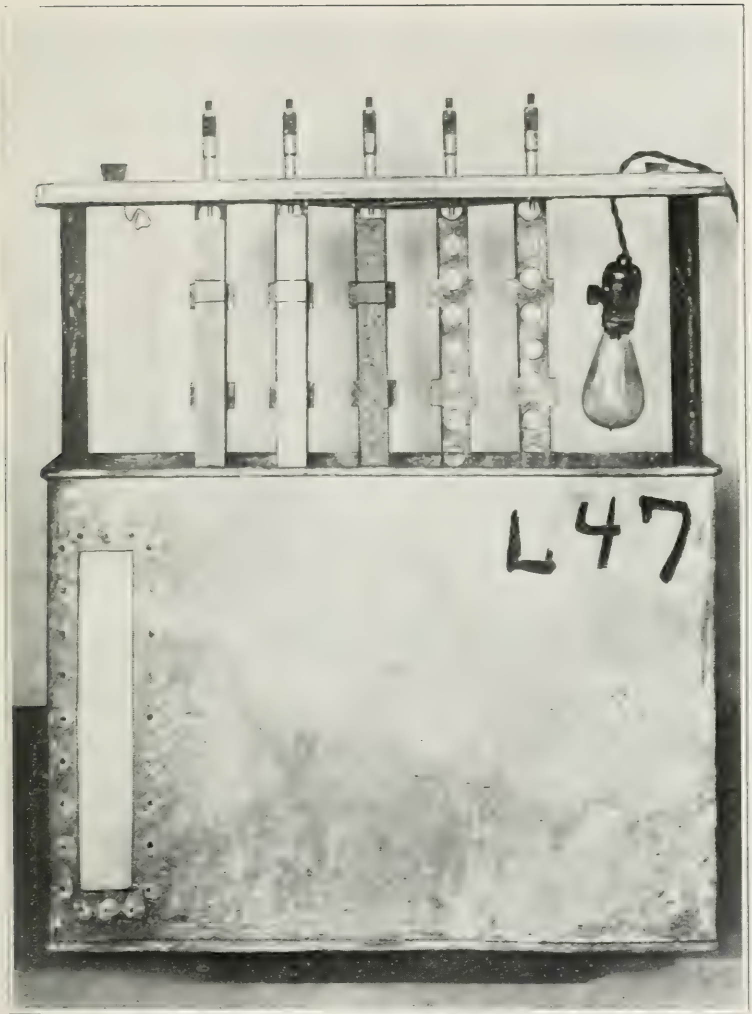

PLATE I.—Apparatus Used for Making Expansion Tests

THE PHYSICAL PROPERTIES OF MAGNESIA CEMENT AND MAGNESIA CEMENT COMPOUNDS

INTRODUCTION

Purpose and Scope of the Bulletin—This bulletin presents the results of an experimental study of the physical properties of magnesia cement and magnesia cement compounds, and of the factors affecting these properties. An important object of the investigation was the determination of physical tests which could be relied upon to indicate the suitability or unsuitability of particular cements or compounds for use as flooring-material. It was also hoped that information would be secured as to the correct proportioning and mixing of the materials for any given purpose, and that representative values for the more important physical constants would be determined.

A description of magnesia cement, its manufacture and uses, is given, together with a summary of the results of chemical investigations.

Acknowledgment—The investigation was begun in 1912 by the Department of Mechanics of the University of Wisconsin, at the suggestion of Mr. William Baumbach, president of the American Monolith Company of Milwaukee. The work was conducted at different times by different individuals. All of the preliminary work, including the tests reported in Tables II, III, and IV, was done by Mr. H. E. Pulver, formerly instructor in mechanics at the University of Wisconsin. The tests reported in Tables V, VI, VII, VIII, XI, and XVI were also made under Mr. Pulver's direction. The tests reported in Tables IX, XII, and XIV were made by Mr. F. D. Lohr under the direction of the writer, in connection with a thesis for the degree Bachelor

of Science, Chemical Engineering Course. The discussion of the chemical nature of magnesia cement, pages 13 to 15, inclusive, and the greater part of the bibliography are taken from Mr. Lohr's thesis.

All the materials used were furnished by Mr. Baumbach, to whom the investigators are also indebted for service and suggestions as to the manner of conducting the tests, and for information concerning the use of the material and the difficulties confronting the manufacturer.

MAGNESIA CEMENT

Definition—Magnesia cement is the product resulting from an admixture of magnesium chloride, water, and properly prepared magnesium oxide. It was invented by Stanislaus Sorel, a Frenchman, in 1853; it is sometimes called Sorel cement, after the inventor, and sometimes magnesium oxychloride cement, on account of its supposed composition.

The powdered magnesium oxide alone is frequently called magnesia cement, and for convenience this usage will be followed in this bulletin wherever no likelihood of confusion is apparent.

Chemical Nature—Sorel[1], though he discovered magnesia cement, did no work on its composition. He merely announced it to be a magnesia oxychloride cement, saying that the hardness varied with the strength of the magnesium chloride solution used, and that solutions ranging in density from 20° to 30° (Baumé) were the most suitable.

Bender[2] in 1870 was the first to publish anything on the chemical composition of magnesia cement. He made tests on six months old specimens of magnesia cement, which had been stored in air. In making his determinations, he considered that the carbon dioxide which had been absorbed from the air had united with the magnesium oxide to form the normal anhydrous carbonate, the rest of the magnesium oxide was considered as being

[1] *Comptes Rendes*, vol. 65, p. 102.
[2] *Annalen der Chemie*, vol. 159, p. 341.

free to react with the magnesium chloride. He extracted the excess magnesium chloride with cold water, assumed the unextracted magnesium chloride to be combined with the free magnesium oxide and computed his composition formulae for the cement. For the six months, air stored Sorel cement he found it to be $MgCl_2.5MgO.17H_2O$. When this was treated with water and dried over SO_3 it lost some of its water, and a compound of the formula $MgCl_2.9MgO.24H_2O$ remained. When the original compound was treated with boiling water and dried over SO_3 there was left a compound of the formula $2MgO.3H_2O$. All of these compounds were hard, and could not be disintegrated by boiling water. He also made a series of tests on the amounts of water lost at different temperatures by these three compounds, and obtained the results shown in the following table:

WATER LOST AT DIFFERENT TEMPERATURES

$MgCl_2.9\ MgO.17\ H_2O.$	heated to	100°C,	lost	9 mol. of water.
" " "	" "	180°C	"	11 " " "
$MgCl_2.5\ MgO.24\ H_2O.$	" "	100°C	"	9 " " "
" " "	" "	180°C	"	14 " " "
$2\ MgO.3\ H_2O$	" "	100°C	"	2.1 per cent of water
" "	" "	180°C	"	4.3 " " " "

Krause[3] in discussing Bender's work claimed that Bender's assumption of the formation of the normal anhydrous carbonate was incorrect, holding that because of the presence of an excess of the basic material it was much more probable that the basic carbonate would be formed. The formation of the basic carbonate would make Bender's results inaccurate.

Krause attempted to prevent the formation of the basic carbonate by the use of an excess of magnesium chloride solution. He treated the magnesium oxide with an excess of concentrated magnesium chloride solution, heated, shook, cooled, and dried this, and obtained a solid of the composition $MgO.MgCl_2.16H_2O$.

Nuhman[4] separated a compound of the composition

$$Mg\begin{matrix}OH\\Cl\end{matrix}.\ 5\ Mg(OH)_2$$

[3] *Comptes Rendes*, vol. 94, p. 444.
[4] *Chemicher Zeitung*, vol. 25, p. 96.

He found that on long treatment with boiling water some of the magnesium chloride went into solution, leaving a chloride of the composition of Brucite.

In 1909 Robinson and Waggeman[5] published the results of a series of tests of mixtures of different concentrations of magnesium oxide and magnesium chloride. These mixes were allowed to stand for six months at a temperature of 25° C. and then the liquid and precipitate were tested. They found the normal composition to be $3MgO.MgCl_2.10H_2O$.

In 1908 and 1909 there appeared a series of articles in the *Chemicher Zeitung* giving the results of several German researches on the composition of the magnesium oxychlorides.

Dr. Van Hoff[6] electrolyzed a concentrated solution of magnesium chloride and obtained at the cathode a white solid to which he gave the formula $MgCl_2.5MgO.14H_2O$. On drying this over sulphuric acid it lost three molecules of water, and on being dried to 180° C. it lost eight molecules of water. He also made a mix of magnesium oxide and concentrated magnesium chloride and treated this vigorously with carbon dioxide. In this way he hoped to get results in a short period of time comparable with Bender's, in obtaining which the cement had been exposed to the carbon dioxide in the air for six months. He removed the excess magnesium chloride by washing with absolute alcohol and computed the formula to be $MgCl_2.5MgO.13H_2O$.

Kallenuer[7] objected to the analyses of the cement by treatment with cold water, which was the method that had been employed by Bender, and by Van Hoff in several of his determinations. He claims that there is no doubt but that some of the combined magnesium chloride as well as the free chloride, is removed by cold water. For this reason the results obtained by considering all of the magnesium chloride removed by the water as being uncombined are of little value. He concludes, from the fact that the chloride can be completely removed from the cement, leaving a hard residue, and from thermo-chemical data, which he was not ready to give at present, that the cement is not an oxychloride at all, but is $Mg(OH)_2$ with magnesium chloride in solid solution.

[5] *Journ. of Phys. Chem.*, vol. 13, p. 673.
[6] *Chemicher Zeitung*, vol. 33, p. 693.
[7] *Chemicher Zeitung*, vol. 33, p. 871.

Krieger[5] treated five grams of magnesium oxide with 200 c. c. of magnesium chloride solution of a specific gravity of 1.25, at room temperature. In two days' time this gave a gelatinous mass. The excess chloride was removed by repeated extractions with absolute alcohol, and the residue dried at 40° C. At the end of different time periods the mass was analyzed and the following results were obtained:

Time	$MgCl_2$	MgO	H_2O
2 days	1 part	5.0 parts	16.1 parts
2 weeks	1 part	3.3 parts	10.2 parts
4 weeks	1 part	2.0 parts	8.9 parts

At the end of four weeks' time, the composition of the compound remained constant, the formula being $MgCl_2.2MgO.9H_2O$. Krieger decided that since there were so many things which affected the cement, such as temperature, proportions of the ingredients, time of reaction, etc., that no set formula could be given for Sorel cement. From the formulae given he evidently considered it to be an oxychloride of magnesia, and not a solid solution of magnesium chloride in $Mg(OH)_2$, as suggested by Kallenuer.

From the researches mentioned, it would seem that Sorel cement is a magnesium oxychloride, having several molecules of water of crystallization. The exact proportions of the compound seem to depend on the conditions to which the cement has been exposed. It seems to have a formula approximating $MgCl_2.3 MgO.10 H_2O$.

Manufacture—The magnesium oxide used for cement is made by calcining and grinding magnesite ($MgCO_3$), a mineral found in fairly extensive and comparatively pure deposits in many parts of the world. The most important foreign sources of supply are in Greece, Austria, Italy, Norway, and South Africa. In this country the most important deposits are in California. The form of the deposit varies; sometimes the material occurs in extensive veins, sometimes as local outcrops, and sometimes, as in eastern Canada, as loose surface rock. In general, however, it must be mined, though this can often be

[5] *Chemicher Zeitung*, vol. 34, p. 246.

done in open-cut; and at some places, notably in Greece, the simplest methods of hand labor are employed.

After being mined, the magnesite is broken into medium sized lumps and calcined in vertical kilns to drive off the carbon dioxide. The material is introduced at the top of the kiln and passes through to the bottom, where it is removed. About twenty-four hours are necessary for thorough calcination, two of which are required for the magnesite to pass through the zone of greatest heat. The weight of carbon dioxide driven off is theoretically equal to half the weight of the magnesite introduced, but owing to loss through over-burning and under-burning, and to a small amount of water usually present, about two and one-third tons of natural magnesite are required to produce one ton of the finished product.

The temperature of calcination is of great importance. It has been shown that carefully prepared and purified $MgCO_3$ gives off all its carbon dioxide when heated to 510° C.[9] The conditions are somewhat different in the case of the natural mineral, however, and it is customary to calcine magnesite intended for cement at a temperature between 700 and 900° C. the exact temperature depending largely on the amount and nature of the impurities present. These impurities consist mainly of lime, silica, iron and aluminum oxides, and sulphur. It is chiefly the iron oxide that governs the burning, a large iron content requiring a low temperature, and a low iron content a high temperature.[10] By far the greater portion of the magnesium oxide produced is used as refractory material for fire brick, furnace and crucible lining, etc. When intended for such use, the magnesite is calcined at a temperature of from 1500 to 1700° C., and the magnesium oxide so produced, known as dead-burned magnesia, to distinguish it from the lightly calcined or caustic magnesia burned at the lower temperatures, is worthless for cement. The fact that this dead-burned magnesia is more important commercially than the other material, has militated against a standardization of production of the latter. For this reason considerable difficulty is experienced by manufacturers in obtaining a

[9] *United States Geological Survey Bulletin*, No. 355, p. 10.
[10] *The Engineer (London)*, vol. 119, p. 471.

uniformly good product, neither underburned nor overburned, and containing a high per cent of magnesium oxide.

The calcined magnesite is finely ground; a fineness such that 10 per cent or less will be retained on a 120 mesh sieve is considered desirable. Grinding is best done with mill-stones, though tube mills have been used.

Magnesium chloride is obtained mainly as a by-product in the manufacture of other substances. At salt works it is made from the bittern water, or refuse left in the evaporating pans after the salt has crystallized out. Most of that which is produced in this country is made in Michigan and in California.

In Germany it is made from "Carnallite" ($MgCl_2.KCl.6H_2O$), a mineral occurring in extensive localities near Strassfurt. The last liquor from the KCl extraction from crude Carnallite, at a specific gravity of 1.32, is evaporated to a specific gravity of 1.34, when the chlorides of sodium and calcium, together with the magnesium sulphate, separate from the solution. The remaining liquid is run into flasks, and on cooling solidifies into a white, translucent mass which is about 50 per cent magnesium chloride.

In applying magnesia cement, it is customary to combine the materials by first mixing the magnesium oxide and dry aggregate thoroughly, and then adding the magnesium chloride in the form of a solution. A solution of almost any strength will produce setting,—indeed, a slight set may sometimes be obtained with water alone,—but a solution having a density of about 1.20 (24° Baumé) is generally used. It is the practice of a few manufacturers to make a semi-fluid mix of the cement and the solution, and then add the aggregate. The magnesium oxide and magnesium chloride may also be combined in the desired proportions dry, the resulting mix requiring only the addition of the aggregate and water. This is the practice at the magnesite mines at Malelane, South Africa, where the magnesite is mined, calcined, ground, mixed with magnesium chloride imported from Germany, and shipped ready for use.[11]

Uses of the Cement—The first use made of magnesia cement was in the manufacture of artificial building stone, called Sorel

[11]*United States Geological Survey Bulletin*, No. 355, p. 13.

stone. This material, made by combining the cement with stone dust, was strong and hard, but proved unable to withstand the effects of the weather, and was therefore not a success. More or less use has been made of the cement since that time. Other attempts have been made to use it in the manufacture of building stone, and it has also been employed as a binding material for emery wheels and grindstones, but its chief use at present is in the manufacture of artificial marble for interior decoration, and, in combination with various inert aggregates, as a composition for walls, wainscoating, and especially flooring. The cement may be used with almost any sort of material as aggregate. Stone-dust, sand, powdered cork, talc, clay, asbestos, and sawdust are all used, depending on the purpose and choice of the manufacturer. Asbestos and sawdust are, however, most generally employed for flooring. Mineral pigments may be used to secure various colors, though most floors are made in different shades of grey or buff.

The composition flooring can be laid directly on concrete, wood, or metal. It is customary to employ two coats or layers of the composition, the first of rather coarse fiber and the second of very fine fiber. If laid on absorbent material, such as brick, the latter must first be thoroughly wet down with the magnesium chloride solution to prevent absorption of liquid from the mix before setting can take place. The total thickness is generally about half an inch. The finishing is best done by hand trowelling; burnishing and scraping are sometimes resorted to, but it appears to be the opinion of most manufacturers that this produces a more absorbent floor and one in which efflorescence is more likely to occur.

The nature and proportions of the ingredients and the manner of mixing and laying the composition differ with the different manufacturers. Owing to this lack of standardization of either materials or methods, specifications are not as yet available, and reliance must be placed on the skill and honesty of the manufacturer. When the proper materials are used, and the work is well done, a most satisfactory floor can be had, one which combines to a remarkable degree resilience, warmth, and quietness, with fire-proof and sanitary qualities. On the other hand, failures sometimes occur, even when the work is skilfully

and conscientiously done, and no manufacturer can claim infallibility. There is a marked and acknowledged need for a standardization of materials and processes, so that architects and engineers shall be enabled to make rational specifications, and so that the manufacturer may be assured of consistent results from careful work. It was as a contribution to the work of such standardization that the investigation reported in this bulletin was undertaken.

MATERIALS USED IN MAKING TESTS

The materials used consisted of cement, magnesium chloride, sand, sawdust, asbestos, and coloring matter.

Cement—In all, twelve different cements were used in the various tests. The chemical analyses of these cements are given in Table I. A description of the method of analysis is given at the end of the bulletin.

It was found impracticable to make complete sieve analyses of the cements, as the material caked in the finer sieves and could not be removed.

The cement was stored in airtight friction-top cans in the laboratory, where the temperature averaged 70° F. and the relative humidity averaged 60 per cent.

The Magnesium Chloride—The magnesium chloride used in the earlier work was received in the form of a solution, having a specific gravity of about 1.22. Later, the magnesium chloride was received in solid form and made into a solution of the desired strength with distilled water.

Only one of the two lots of magnesium chloride received was analyzed. Analysis showed that the crystals contained 46.9 per cent $MgCl_2$. According to the formula ($MgCl_2.6H_2O$) such crystals should contain 46.78 per cent $MgCl_2$ and 53.22 per cent water. There was no calcium present in this sample, and but a trace of sulphate. It is believed that the sample analyzed may be regarded as representative of all the magnesium chloride used, the source of supply being the same, and hence that this material may be considered to have been practically pure.

The Sawdust—The sawdust used was a hard maple sawdust containing about 8 per cent of moisture. A mechanical analysis of this sawdust was made; the curve obtained is shown in Fig. 1.

The Asbestos—The asbestos used was a very finely shredded asbestos fiber.

The Sand—The sand used was the standard Ottawa sand specified for use in the standard commercial test of Portland cement. This is a white silica sand of uniform quality, screened to pass a No. 20 sieve and to be retained on a No. 30 sieve.

The Coloring Matter—The coloring matter used was iron oxide, Fe_2O_3 in powdered form.

TABLE I

CHEMICAL COMPOSITION OF THE CEMENTS USED

Cement	Per cent of SiO_2	Per cent of FeO_3 & AlO_3	Per cent of CaO	Per cent of MgO	Per cent of SO_2	Per cent loss on ignition
A	7.84	1.24	1.62	77.87	0.0	11.90
B	0.68	0.66	19.07	66.45	1.31	12.82
C	2.08	0.61	3.00	81.03	0.36	12.84
D	3 88	0.42	17.08	56 11	0.0	21.91
E	3.39	0.70	2.67	86.78	0.0	6.79
F	4.84	0.46	3.58	86.44	0.0	4.95
G	3.52	0.66	2.97	86.37	0.24	6.25
H	2.74	0.28	3.03	91.76	Trace	1.57
I	5.14	0.32	4.76	82.95	Trace	6.07
J	11.51	1.04	0.73	60.81	....	....
K	3.96	0.37	0.05	63.25	....	....
L	5.18	0.47	9.09	54.02	....	....

PRELIMINARY TESTS

Purpose—Before commencing the tests intended to show the effect of different variables, it was necessary to formulate certain standards according to which the work should be carried on. It was considered desirable to determine:

(1) The best method of mixing and molding the specimens;

(2) The usual time of set;

(3) The best mix to be used for the tests;

(4) The best cement to be used for the tests;
(5) The best storage medium;
(6) The best age at which to test the specimens;
(7) The comparableness of tests made at different times, on specimens made at different times, by the same operator;
(8) The comparableness of tests made at the same time, on specimens made at the same time, by different operators.

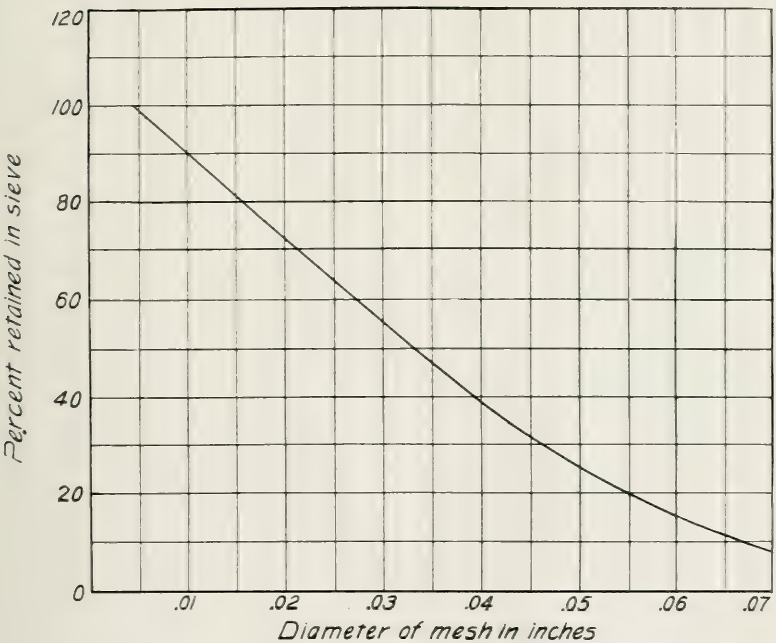

FIGURE 1.—Mechanical Analysis of Sawdust

Discussion of Tests and Results—After some experimenting, it was found that a slight modification of the method of mixing and molding specified for the standard commercial test of Portland cement gave satisfactory results for this material[1]. In mixing, the dry materials were first thoroughly mixed in an enamelled vessel, and the prescribed amount of magnesium chloride solution then added. After all the liquid had been

[1] Standard Specifications and Tests for Portland Cement. *American Society for Testing Materials Standards*, 1916, p. 429.

thoroughly incorporated, the mixture was vigorously kneaded by hand for 1½ minutes, and the material was placed in the molds in layers, pressed down firmly with the thumbs, and troweled smooth, but not rammed. It was found that excessive troweling brought the liquid to the surface, so no more of this was done than seemed necessary.

The time of set was determined by means of the Vicat needle, the method being the same as that used in determining the time of set of Portland cement. All tests were made on neat cement mixes of standard consistency, as determined by the Vicat needle, but different densities of solution were employed. The time of initial set was found to be from one to three hours; the time of final set was found to be from three and one-half to seven hours. No consistent relation was found to exist between time of set and density of solution, but the effect of temperature was evident, a low temperature retarding set to a very marked degree.

There was considerable uncertainty as to the best mix for use in making the tests. The number of materials which have been successfully employed as aggregates in combination with magnesia cement is very great, and it was evident that the properties of the compound must depend, to a considerable extent, on the nature and relative amount of the aggregate. Moreover, most of the materials commonly used for this purpose are of such a nature as to render standardization difficult. It was thought, therefore, that tests on a neat mix, or on a mix made up with an aggregate of uniform and known nature, might be more valuable than tests made on compounds of the sort ordinarily used. It was therefore decided to use neat cement and a 1:3 mortar of the standard Ottawa sand already described. At the same time it was seen to be necessary to determine whether the effect of the different variables would be the same on these experimental mixes as on the compounds used in practice, and this could only be done by accompanying the tests made on them by tests made on some one of the latter.

A mix had been suggested by the manufacturer as one giving good results in floor construction. This mix consisted of cement, sawdust, asbestos, and coloring matter in the following proportions by weight: cement, 49.3 per cent; sawdust, 29.2 per cent; asbestos, 13.4 per cent; coloring matter, 8.1 per cent. As

sawdust and asbestos are perhaps the materials most commonly employed as aggregates in floor construction, it was believed that this mix could be regarded as a compound typical of those used in practice, and it was accordingly adopted for testing. It will hereafter be referred to as the "standard floor mix".

It was later decided to dispense with the coloring matter, and use a mix consisting of cement, sawdust, and asbestos, the proportion of these constituents being kept the same as before. This gave a mix containing 53.7 per cent cement, 31.7 per cent sawdust, and 14.6 per cent of asbestos. Comparisons of results showed that the omission of coloring matter had no effect, and in the following discussion no distinction will be made between the colored and the uncolored mixes, the term "standard floor mix" being used with reference to both.

It was suggested by the manufacturer that the magnesium chloride solution be added to the standard floor mix in the proportion of 85 cubic centimeters of solution to every 100 grams of solid material. The consistency of the mix so made was considered to be standard, and the mixes made up with sand were considered to be of standard consistency when, as nearly as could be judged, they were the same.

It was decided to use all the three mixes described above,—the neat, the 1:3 sand, and the standard floor mix,—in a sufficient number of tests to determine whether the three were similarly affected by different variables, and to determine which yielded the most uniform and consistent results.

In a series of tests intended to show the effect of any one variable, it was desired to use that one of the several cements from which the most consistent results were to be expected. It was also considered important to store the specimens during curing in such a way and to test at such an age as to secure representative and, as nearly as possible, uniform results. In order to obtain information on these points, an extensive series of tension tests was planned. Specimens were made up of the three mixes and of the three cements at that time on hand,—D, E, and F. It was known that moisture in the atmosphere was supposed to exert a disintegrating effect on magnesia cement, and the methods of storage were chosen with a view to securing data on this effect. Five storage mediums were employed; namely, moist air,

having a humidity of 90 per cent; air in the laboratory (hereafter called "normal air"), having an average humidity of 60 per cent; dry air in a dessicator; a solution of $MgCl_2$ and kerosene. Specimens were broken at seven, fourteen, and twenty-eight days.

The results of this series of tests are given in Table II. Each individual value for tensile strength represents the average strength of four similar specimens made and tested at the same time, and each corresponding value for per cent variation from the mean represents the average variation, from this average, of the four specimens.

To facilitate a comparison of the results obtained with the different methods of storage, the diagrams of Fig. 2 and Fig. 3 have been drawn. These diagrams were obtained by plotting at equal horizontal intervals the average of all tensile strengths and all variations from the mean for each mix and each method of storage.

The results of this series of tests show:

(1) That of the three mixes tested, the standard floor mix gave by far the most uniform results for all cements, all ages, and all methods of storage.

(2) That of the five methods of storage, storage in kerosene and in normal air gave the most satisfactory results.

(3) That the ranking of the different cements as to strength and uniformity was different for different mixes and at different ages.

(4) That the relative strength of the specimens at different ages was different for the different mixes,—the strength decreasing with age in the case of the neat and 1:3 sand mixes, but increasing with age in the case of the floor mix.

(5) That in general the per cent departure from the mean increased as the age of the specimens increased.

A second series of tests was made to serve as a check on some of the above results, to determine the desirability of machine oil as a storage medium, and to determine the comparableness of results obtained by different operators working at the same time. Specimens were made up of the same three mixes and

DETERMINATION OF THE BEST STORAGE MEDIUM AND OF THE BEST MIX, BEST CEMENT, AND BEST AGE FOR TESTING
RESULTS OF TENSION TESTS

Mix	Cement	Age Days	Method of Storage: *In Moist Air* Tensile str'ngth Lb. per Sq. In.	*In Moist Air* Per Ct. Variation from Mean	*In Normal Air* Tensile str'ngth Lb. per Sq. In.	*In Normal Air* Per Ct. Variation from Mean	*In Dry Air* Tensile str'ngth Lb. per Sq. In.	*In Dry Air* Per Ct. Variation from Mean	*In Mg Cl_2 Solution* Tensile str'ngth Lb. per Sq. In.	*In Mg Cl_2 Solution* Per Ct. Variation from Mean	*In Kerosene* Tensile str'ngth Lb. per Sq. In.	*In Kerosene* Per Ct. Variation from Mean	Average at Each Age for All Methods of Storage: Tensile strength Lb. per Sq. In.	Average at Each Age: Per cent Variation from Mean	Average for All Ages for All Methods of Storage: Tensile strength Lb. per Sq. In.	Average for All Ages: Per cent Variation from Mean
Neat	D	7	625	4.8	757	10.8	980	2.5	268	5.7	985	6.2	723	6.0		
		14	440	14.4	647	4.8	751	10.5	185	5.5	933	7.5	591	8.5	633	11.7
		21	326	7.8	703	7.1	815	61.0	116	21.1	959	5.4	584	20.5		
	E	7	14	98.3	432	19.1	494	16.1	14	98.2	684	13.9	328	49.1		
		14	15	146.7	299	45.7	662	22.8	6	145.8	486	13.6	293	74.9	296	67.5
		21	24	148.0	396	49.9	125	99.8	0	....	787	16.6	266	78.5		
	F	7	905	9.9	1029	3.6	1110	....	356	8.1	1110	....	902	7.2		
		14	750	22.9	842	10.4	1107	0.5	182	7.8	1087	3.2	794	8.9	781	14.9
		21	463	21.6	1002	5.7	627	70.4	29	45.7	1107	0.4	646	28.7		
	Average....		395	52.7	678	17.4	741	31.5	128	42.1	904	7.4	570	31.0		
1:3 Sand	D	7	190	13.0	363	5.6	379	5.7	39	48.7	266	4.8	247	15.5		
		14	233	2.4	369	5.2	284	10.4	34	29.4	300	5.6	244	10.6	233	12.6
		28	108	10.4	229	18.5	339	12.5	88	12.8	276	4.1	208	11.6		
	E	7	449	8.2	703	3.1	742	4.3	92	10.6	740	2.8	545	5.8		
		14	203	10.5	558	3.2	362	13.7	74	16.5	724	4.5	384	9.6	442	7.9
		28	139	12.4	434	2.8	531	11.6	97	10.6	786	4.1	397	8.3		
	F	7	191	13.9	388	6.6	581	4.7	85	2.6	365	10.7	322	7.7		
		14	127	12.2	361	4.2	734	3.3	61	8.2	366	4.9	329	6.5	290	9.0
		28	103	4.1	278	16.9	188	14.8	90	20.0	437	8.1	219	12.8		
	Average ...		193	9.6	409	7.3	460	9.0	73	17.7	471	5.5	322	9.8		
Standard Floor Mix	D	7	265	8.2	283	2.9	270	6.0	277	5.0	262	4.9	271	5.4		
		14	290	2.3	329	0.8	308	5.6	272	4.6	261	6.0	292	3.8	281	4.8
		28	266	3.3	282	1.8	368	8.5	254	5.3	238	6.9	281	5.2		
	E	7	435	5.8	510	7.1	478	3.3	454	3.2	443	2.2	464	4.3		
		14	434	5.8	506	8.5	505	5.8	454	3.4	415	3.7	462	5.4	478	5.4
		28	443	5.8	532	6.5	672	9.1	481	6.4	409	4.2	507	6.4		
	F	7	452	7.0	494	4.1	471	4.3	427	2.6	424	4.3	453	4.4		
		14	479	2.9	561	2.6	513	9.9	446	5.1	422	2.9	484	4.6	488	5.0
		28	434	3.1	584	5.1	760	8.1	458	6.1	395	7.7	526	6.0		
	Average ...		389	4.9	453	4.4	482	6.7	391	4.6	363	4.7	416	5.1		

with the same three cements that were used before, but only three methods of storage were employed; namely, storage in normal air, in kerosene, and in machine oil. Specimens were broken at seven and at twenty-eight days.

The results of this series of tests are given in Table III, and shown graphically in Fig. 2 and Fig. 3. As before, each strength value represents the average for four specimens, and each value for per cent departure from the mean the average departure for those four. The results presented show:

(1) That of the three mixes tested the standard floor mix gave by far the most uniform results for all cements, all ages, and all methods of storage.

(2) That of the three methods of storage, storage in oil gave the best results with the neat briquettes, storage in kerosene with the 1:3 sand, and storage in normal air with the floor mix.

(3) That the ranking of the different cements as to strength and uniformity was different for the different mixes and at different ages.

(4) That for all mixes the strength increased with age.

(5) That the per cent departure from the mean was practically the same at the two ages.

(6) That the results obtained by the two operators, both as regards strength and uniformity, varied widely in the tests made on the neat cement and agreed closely in the tests made on the floor mix, and the 1:3 sand mix.

The tests to compare methods of storage showed that the strength of specimens might be considerably affected by conditions of humidity and temperature during storage, and suggested that such might also be the case as to conditions during making. It seemed probable that tests made at different times, on specimens made at different times, would not yield comparable results, even though the work were done by the same operator and similar materials were used.

To secure information on this point, specimens were so made up and tested. The results are given in Table IV.

It will be seen that a very considerable difference exists between the results of tests made on different dates, not only as to the actual values obtained, but also as to the variation of strength with age.

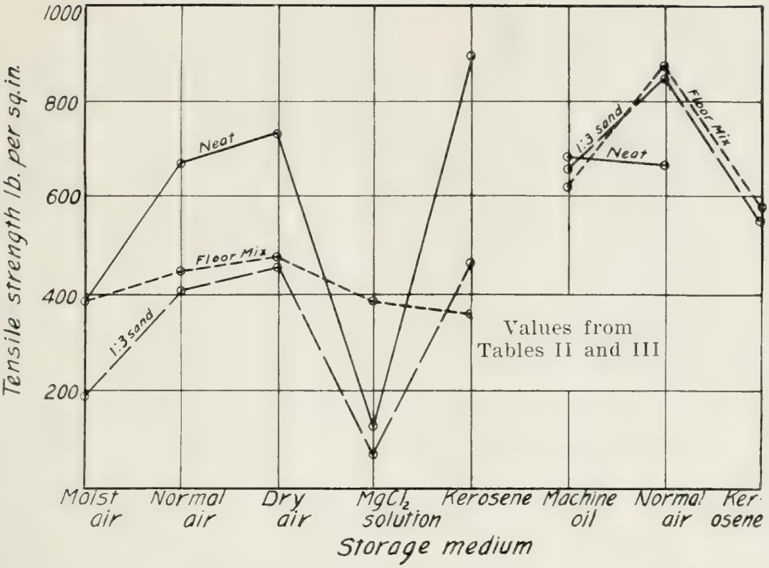

FIGURE 2.—Effect of Storage Medium on Tensile Strength

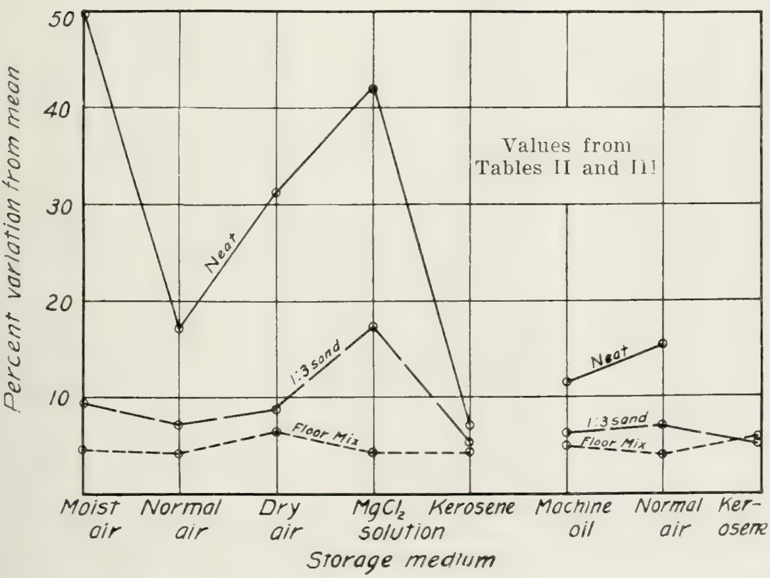

FIGURE 3.—Effect of Storage Medium on Uniformity of Tensile Strength

TABLE III

DETERMINATION OF THE BEST STORAGE MEDIUM, OF THE BEST MIX, BEST CEMENT, AND BEST AGE FOR TESTING, AND OF THE COMPARABLENESS OF TESTS MADE BY DIFFERENT OPERATORS

RESULTS OF TENSION TESTS

Mix	Cement	Age Days	Method of Storage: In Machine Oil — Tensile Strength Lb. per Sq. In. — Operator X	Y	Av.	In Machine Oil — Per Cent Variation from Mean — Operator X	Y	Av.	In Kerosene — Tensile Strength Lb. per Sq. In. — Operator X	Y	Av.	In Kerosene — Per Cent Variation from Mean — Operator X	Y	Av.	In Normal Air — Tensile Strength Lb. per Sq. In. — Operator X	Y	Av.	In Normal Air — Per Cent Variation from Mean — Operator X	Y	Av.	Average at Each Age for All Methods of Storage and Both Operators — Tensile Strength Lb. per Sq. In.	Perce't Variation from Mean	Average for Both Ages for All Methods of Storage and Both Operators — Tensile Strength Lb. per Sq. In.	Perce't Variation from Mean
Neat	D	7	782	757	769	6.1	3.0	4.6	......	......	......	......	......	......	1017	1039	1028	7.1	4.7	5.9	899	5.3	898	7.1
		28	724	689	707	7.5	6.8	7.2	......	......	......	......	......	......	1163	1006	1085	3 1	18.1	10.6	896	8.9		
	E	7	360	583	472	21.0	8 9	14.9	......	......	......	......	......	......	126	340	233	11.1	35.4	23.3	353	19.1	405	20.9
		28	322	613	468	16.2	28 6	22.4	......	......	......	......	......	......	666	2[illegible]0	443	31.3	14.3	22.8	456	22.6		
	F	7	967	805	886	6.3	15.3	10 8	......	......	......	......	......	......	561	507	534	15.5	11.0	13.3	710	12.1	748	13.4
		28	876	802	839	14.4	6.5	10 5	......	......	......	......	......	......	709	751	730	7.9	29.2	18.6	785	14.6		
	Average for all Cements		672	708	690	12.1	11.5	11.8	......	......	......	......	......	......	707	644	676	12.7	18.8	15.7	......	......	684	13.8
1:3 Sand	D	7	367	341	354	1.4	12.4	6.9	380	297	339	7.1	15.2	11.2	830	756	793	9.8	5.1	7.5	495	8.5	516	9.0
		28	361	399	380	6 9	12.3	9.6	375	436	406	6.9	4.3	5.6	814	827	821	5.8	20.2	13.0	536	9.4		
	E	7	818	855	837	7 7	7.1	7.4	705	761	733	2.2	3.8	3.0	796	752	774	7.3	3.8	5.6	781	5.3	857	4.9
		28	964	976	970	1.1	5.5	3.3	800	832	816	3.4	4.9	4.2	1025	1003	1014	5.4	6.4	5.9	933	4.5		
	F	7	706	607	657	2.2	7.9	5.1	553	463	508	2.9	6.5	4.7	650	770	710	7.8	6.2	7.0	625	5.6	730	5.1
		28	788	795	792	7.2	5.8	6.5	694	677	686	4.5	2.4	3.5	1027	1030	1029	2.9	4.4	3.7	836	4.6		
	Average for all cements		667	662	665	4.4	8.5	6.5	585	578	582	4.5	6.2	5.3	857	856	857	6.5	7.7	7.1	......	......	701	6 3
Standard Floor Mix	D	7	314	311	313	3.8	4 0	3.9	321	316	319	7.6	7.3	7.5	398	404	401	3.2	5 6	4.4	344	5.3	355	5.4
		28	308	291	300	4.9	9.0	7.0	316	280	298	3.5	6.2	4.9	498	498	498	4 3	4 9	4 6	365	5.5		
	E	7	636	677	657	5 0	4.2	4.6	658	650	654	3.7	9.5	6.6	850	818	834	2.5	3 1	2.8	715	4.7	734	4.3
		28	656	675	666	6.9	1.3	4.1	679	705	692	3.5	4.6	4.1	912	886	899	4.4	2.7	3.6	752	3.9		
	F	7	592	576	584	5.6	4 6	5.1	627	556	592	4 4	8.1	6.3	811	808	810	6.9	4 3	5.6	662	5.7	682	5.4
		28	641	617	629	3.8	7.5	5.7	599	587	593	5.0	6.5	5.8	934	828	881	4.6	3.0	3.5	701	5.0		
	Average for all cements		525	525	525	5.0	5.1	5 1	533	516	525	4 6	7.0	5.8	734	707	721	4.2	4.0	4.1	......	......	590	5.0

TABLE IV

COMPARABLENESS OF TESTS MADE AT DIFFERENT TIMES, ON SPECIMENS MADE AT DIFFERENT TIMES, BY THE SAME OPERATOR

RESULTS OF TENSION TESTS

Mix	Cement	Date of Making Specimen	Amount Mg Cl_2 Solution c. c. per 100 Gm. Solid Material	Tensile Strength Lb. per Sq. In.	
				7 Days	28 Days
Neat	J	1	51	704	604
		2	51	551	1015
	K	1	42	219	662
		2	42	528	785
		3	42	837	775
	L	1	36	963	1015
		2	36	935	915
1:3 Standard Sand	J	1	15	601	555
		2	15	819	1015
		3	15	796	793
	K	1	13.5	750	1015
		2	13.5	966	744
	L	1	12.5	725	938
		2	12.5	631	588
Standard Floor Mix	F	1	85	718	
		2	85	496	
		3	85	634	

Conclusions from Preliminary Tests—On the basis of the results secured from these three series of tests it was decided that, in general, all tests intended to show the effect of variables other than the amount of aggregate should be made on specimens of the standard floor mix; that specimens should be stored in normal air; that tests should be made at any time after three days, and that all tests the results of which were to be compared quantitatively should be made at the same time, on specimens made at the same time and by the same operator.

TESTS TO DETERMINE THE EFFECTS OF DIFFERENT VARIABLES

Variables Considered—In a material such as that under consideration, the number of variables which may affect the properties of the finished product is manifestly very great. The cements ordinarily available differ in respect to the amount of the various impurities contained, the amount of magnesium oxide present, and the proportion of this magnesium oxide that has been properly burned, so as to be active. Practice differs concerning the amount and strength of magnesium chloride solution used, and as to the nature and proportions of the materials used as aggregates. If it be assumed that the magnesium chloride and the magnesium oxide always combine according to some established formula, then it would seem that the cement and solution should be proportioned according to this formula. But there is also to be considered the question of consistency; the mix must be neither too dry nor too wet for convenience and ease in handling. These considerations prompt the questions: should the amount of the magnesium chloride solution be such as to give the theoretically ideal proportion of $MgCl_2$ to MgO, or should it be such as to give the best working consistency? Can both ends be met by varying the strength of the solution inversely as to the amount, in such manner as to at once secure the required amount of $MgCl_2$ and the desired consistency? Can this be done more cheaply by wetting the aggregate in advance of mixing, thus making possible the use of a strong solution of $MgCl_2$ in a smaller proportion than would alone suffice to give normal consistency?

Another question of importance has to do with the proportion of aggregate which should be employed in order to secure the maximum economy consistent with good results.

With a view to covering the above points as thoroughly as possible, it was planned to investigate the effect of:

(1) The chemical constitution of the cement.

(2) Variation in the amount of $MgCl_2$ solution; density constant.

(3) Variation in the density of the $MgCl_2$ solution; amount constant.

(4) Variation in both the amount and density of $MgCl_2$ solution; amount of $MgCl_2$ constant.

(5) Variation in the proportion of cement to aggregate.

(6) Variation in the moisture content of the aggregate.

The Tests to be Employed—It was desired to investigate the effect of these variables in such ways as to determine their practical significance, with particular reference to the use of the material as flooring. In cases where flooring of this kind has failed, the failure has often been due not so much to any lack of strength or hardness as to what may be termed inconstancy of volume; that is, a tendency to expand, shrink, or warp. In general, then, tests which might be relied upon to ascertain the quality of materials whose function is the resistance of stress would not alone be sufficient in this case. At the same time it is evident that tests which would indicate with precision the tendency to shrink or expand would require special apparatus of a rather delicate nature, and would, therefore, not be adapted to general use. It seemed, therefore, desirable to accompany such tests of this sort as might be made by standard and easily made strength tests, in order to see whether or not any relation could be established between the results. It was also thought that changes in volume might be accompanied by corresponding changes in weight, and that if this could be shown to be the case, a determination of the change in weight would serve practically the same purpose as a determination of the change in dimensions, besides being more easily made.

Strength tests were also considered of importance in themselves. There seems to be no reason why a flooring material, not ordinarily subjected to tension, should be required to show any stated tensile strength. At the same time, if it were found that satisfactory material ordinarily had a tensile strength above a certain figure, then a marked falling below this value would provide grounds for suspecting the quality of the sample in question. For instance, one of the greatest sources of difficulty in the securing of uniform results with composition flooring is the presence of over-calcined or dead-burned magnesite in the cement. Chemical analysis, as ordinarily made, fails to distinguish

between the active or caustic magnesite, and this over-calcined material, which is, of course, worthless. Tensile tests, it would seem, could be relied upon to detect the presence of any considerable quantity of this inert material, at least in a magnesite the normal strength of which had been determined.

The fact that the resistance of flooring to compressive stresses, especially stresses of high intensity occuring over small areas, is often of great importance, seemed to warrant the adoption of compression and hardness tests. The value of cross-breaking tests was also apparent, not only to furnish data as to the flexural strength of the material, but also to give information as to its flexibility, or capacity to sustain enforced deflection without rupturing.

It was decided, therefore, to employ some or all of the following tests in determining the effect of each of the variables listed above:

(1) Tension test.
(2) Compression test.
(3) Cross-breaking test.
(4) Hardness test.
(5) Expansion test.
(6) Change in weight test.
(7) Pat test.

Description of Tests—The tension test was similar to that made on Portland cement, briquettes being made up in standard gang molds and broken in a Riehlé automatic testing machine.

The compression test was made on 2-inch cubes, which were tested in a Riehlé universal testing machine. A spherical bearing block was used and a speed of 0.05 in. per minute was employed.

The cross-breaking test was made on specimens 1 inch square and 14 inches long, which were tested in a small Olsen beam machine. This machine is operated by hand, the load being applied at the center point of the span by means of a screw. The deflection was measured at the center point of the span, loads corresponding to increments in deflection of 0.02 in., maximum loads, and maximum deflections being recorded. The modulus of rupture was computed by the formula

$$S=18\,P,$$

S being the modulus of rupture in lb. per sq. in. and P being the load in lb. causing rupture.

The modulus of elasticity was computed in all cases for a deflection of 0.04 in., by means of the formula

$$E=10,800\ P$$

E being the modulus of elasticity in lb. per sq. in. and P the load in lb. causing a deflection of 0.04 in.

The hardness test employed was the Brinell hardness test. The operation consists in subjecting the specimen to a certain pressure applied through a steel ball, 10 mm. in diameter. Dividing the load applied by the surface area of the indentation produced gives the unit pressure resisted by the specimen, and this is taken as a measure of the hardness of the material. This test was found to be practicable only in the case of specimens made of the floor mix, and as the machine used was of a type designed for the testing of metals, the results obtained for the softer specimens cannot be regarded as more than approximate.

The form of specimen used for the expansion test was identical with that used for the cross-breaking test, except that in order to provide fixed surfaces to which to measure, small glass plates were set in both ends of the test piece. The test consisted in measuring accurately the change in length of the specimen from time to time under various conditions of storage. The apparatus used is shown in Plate I. It consisted of a galvanized iron box, a lid carrying micrometer screws, and a rack for holding the specimens. This rack, which was attached to the under side of the lid, consisted of a rectangular frame of light angle-irons, carrying vertical channels of galvanized iron designed to hold the specimens securely in place. In the horizontal piece that formed the bottom of the frame, small rivets were set, one in front of each channel, for the lower ends of the specimens to rest upon. The micrometer heads on the lid were mounted directly over the channels, so that they could be screwed down against the glass plates in the upper ends of the specimens and any change in the length of the latter thus determined by direct measurement. The lid of the box was provided with a rubber gasket, so that when closed the box was practically air-tight. This made it possible to secure, by the use of dessicating ma-

terial or water, practically constant conditions of dryness or humidity, and thus study the behavior of specimens under either condition of storage. The front of the box was provided with a glass window, so that by means of a thermometer suspended immediately within, the temperature of the interior could be determined at any time.

The change in weight test consisted in determining the loss or gain in weight experienced by the expansion specimens during the test. In most cases the weighing was performed, not on the expansion specimens themselves, but upon small cylinders, made of the same mix, and placed in the box with the expansion specimens. The pat test was identical with the pat test as performed on Portland cement, except that all pats were stored in normal air. In addition to the formal tests described, expansion specimens were kept and examined from time to time for signs of warping, efflorescence or discoloration.

1—The Effect of the Chemical Constitution of the Cement

Tests of Series A—By testing, in the same manner and at the same time, specimens made with different cements, it was hoped that the influence of the various chemical constituents on the physical properties might be determined. The tests, the results of which are presented in Table V, were made in this way.

Each value given in the columns headed "tensile strength" represents the average strength of eight briquettes, four of which were tested by one operator and four by another. The results of any one set of tests, as Set 1, Set 2, etc., were obtained from specimens made up at the same time and tested at the same time, and only the results of the same set should be compared.

To facilitate a study of the results of this series of tests, the diagrams of Fig. 4 have been drawn. Actual tensile strengths are shown plotted as ordinates on vertical lines corresponding to the several cements. The upper curves indicate the relative amounts of lime, magnesium oxide, iron, and aluminium oxides, contained in the different cements. The actual percentages of these constituents were not plotted, but the average amount of each for all five cements was computed, and the per cent of this average contained by any cement was plotted on the vertical line corresponding to that cement.

TABLE V

Effect of the Chemical Composition of Cement, as Determined by Tests on Different Cements

Results of Tension Tests: Series A

The tests of any one set were made at the same time

Mix	Set No.	Cement	Amount Mg Cl_2 Solution c.c. per 100 gm. Solid Material	Density of Mg Cl_2 Solution	Tensile Strength Lb. per Sq. In. 7 Day	Tensile Strength Lb. per Sq. In. 28 Day	Tensile Strength Lb. per Sq. In. Average both ages
Neat	1	A	51	1.26	936	817	877
		B	64	1.26	50	239	145
		C	47	1.26	940	530	735
	2	D	40	1.26	1118	1074	1096
		E	55	1.26	173	0	87
		F	49	1.26	1072	202	637
	3	D	40	1.20	692	948	820
		E	55	1.20	136	402	269
		F	55	1.20	439	876	658
	4	A	51	1.20	911	872	892
		B	67	1.20	90	299	195
		C	47	1.20	957	922	940
1:3 Sand	5	E	12	1.32	780	895	838
		F	12	1.32	694	652	673
		E	12	1.26	677	853	765
		F	12	1.26	679	749	714
	6	A	12	1.26	599	743	671
		B	12	1.26	199	399	299
		C	12	1.26	650	703	677
		D	12	1.26	372	481	427
	7	A	12	1.32	798	678	738
		B	12	1.32	345	477	411
		C	12	1.32	808	917	863
		D	12	1.32	693	740	717
	8	D	12	1.20	257	542	400
		E	12	1.20	303	745	524
		F	12	1.20	353	679	516
	9	A	12	1.20	574	638	606
		B	12	1.20	50	197	124
		C	12	1.20	410	609	510
Standard Floor Mix	10	A	85	1.15	614	732	673
		B	85	1.15	329	413	371
		C	85	1.15	561	632	597
		D	85	1.15	508	524	516
		E	85	1.15	605	738	672
		F	85	1.15	623	720	672

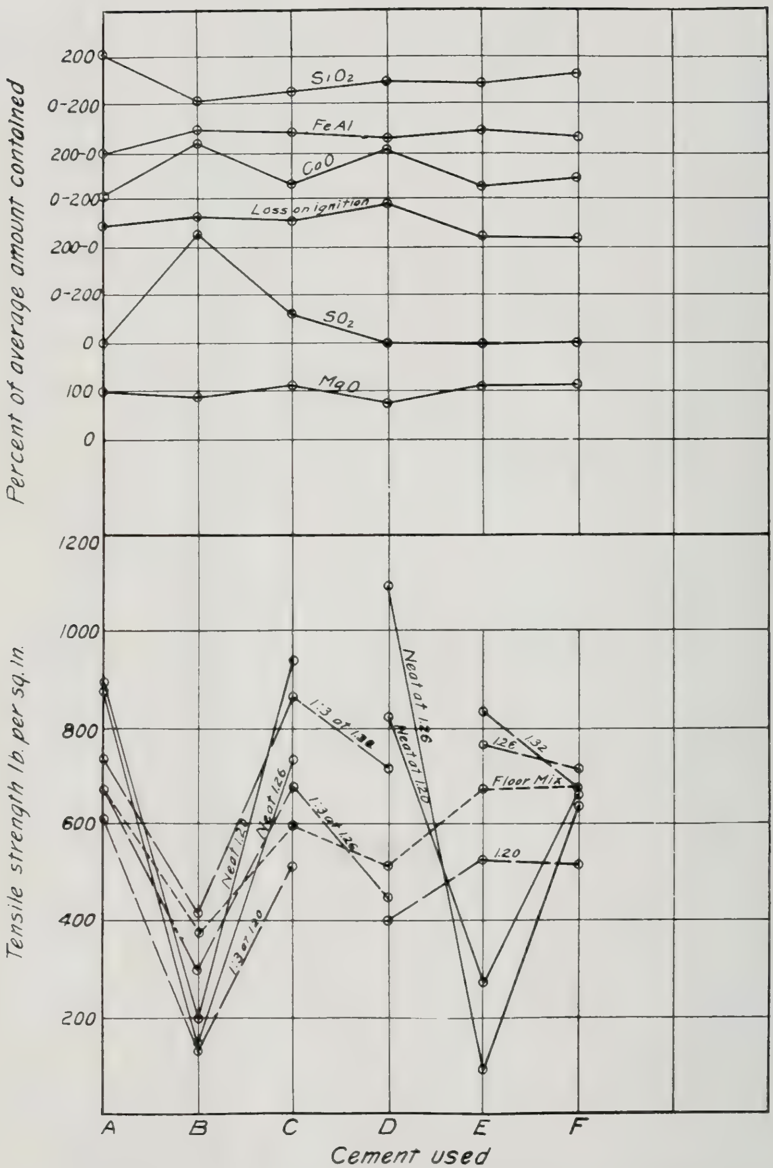

FIGURE 4.—Effect of Chemical Constitution of Cement

A study of these diagrams shows at once that the ranking of the different cements as to strength is the same for all densities of $MgCl_2$ solution. The ranking as determined by tests on neat, 1:3 sand, and standard floor mixes is not, however, the same. For cements A, B, C, and D, the results appear to be consistent, but the relative strength of cements D, E, and F is not at all the same for the neat mix as for the 1:3 sand and floor mixes. If the strength curves are compared with the curves indicating the amounts of chemical ingredients, it is seen that the curves for tensile strength and lime content are opposite in trend, while the curves for strength and MgO content are parallel. It would thus seem that the tensile strength varies directly as the MgO content and inversely as the lime content. No consistent relation seems to obtain between tensile strength and the amount of any other constituent.

Tests of Series B—It would be expected that, other things being equal, the tensile strength would depend directly upon the amount of MgO present in the cement. The reduction in strength accompanying increased lime content might be due either to some deleterious effect of the lime, or to the corresponding reduction in MgO, the lime being, in some cases, present in sufficiently large amounts to appreciably reduce the per cent of MgO. It has, however, been generally considered that lime has a bad effect on magnesia cement. In order to obtain quantitative data on this effect, tests were made on cements to which varying amounts of lime had been added. The results of these tests are given in Table VI. They show clearly the very considerable reduction in strength accompanying the presence of even a small amount of lime.

TABLE VI

EFFECT OF LIME CONTENT, AS DETERMINED BY ADDITION OF LIME
RESULTS OF TENSION TESTS; SERIES B

Cement	Tensile Strength. Lb. per Sq. In.			
	3.70 per cent Lime Added	2.69 per cent Lime Added	1.68 per cent Lime Added	0.67 per cent Lime Added
E	329	364	486	826
F	328	320	470	653

Tests of Series C—In order to determine whether or not any relation obtained between chemical constitution and constancy of volume, expansion tests were performed upon specimens made of four different cements. Two sets of tests were made; in one, the specimens were left exposed to the air of the laboratory; in the other, they were kept in the box under practically constant conditions. The time-expansion curves are shown in Fig. 5 and in Fig. 6. In Fig. 5 is also shown the time-humidity curve for the period of test, and it is seen that the behavior of the specimens is closely related to the humidity of the air, an increase in humidity being accompanied by expansion of the specimens, and a decrease by shrinkage, or at least by a reduction in rate of expansion.

It is seen that the cement which during the first few hours expanded most, later shrank the most. There does not appear to be any very definite relation between the amount of expansion or shrinkage, and the chemical constitution, and the behavior of the different cements in the two sets of tests does not appear to have been consistent.

In addition to the expansion tests described, neat pats were made of cements, A, C, D, and F. Some of these remained perfectly sound for two years, others cracked badly in a few weeks. The cement which made the best showing in this test was cement D. Pats of this cement showed no cracks nor discoloration whatever at the end of two years. Pats of cement F cracked, but did not disintegrate or change color. Cement C turned brown and checked badly, and cement A turned gray and checked very badly, the whole pat breaking up into pieces not more than half an inch square. It does not seem as though the difference in behavior of the various cements in this test could be ascribed to any of the chemical constituents. For instance, cement D, which made the best showing, had a very high lime content, a high loss on ignition, and a very low MgO content. Cement F had a low lime content and high MgO content, but so, too, had cements A and C. In the expansion test, cement D showed somewhat greater constancy of volume, after initial expansion, than cement F.

The results of the expansion tests of this series raised the question of whether or not the change in volume of the speci-

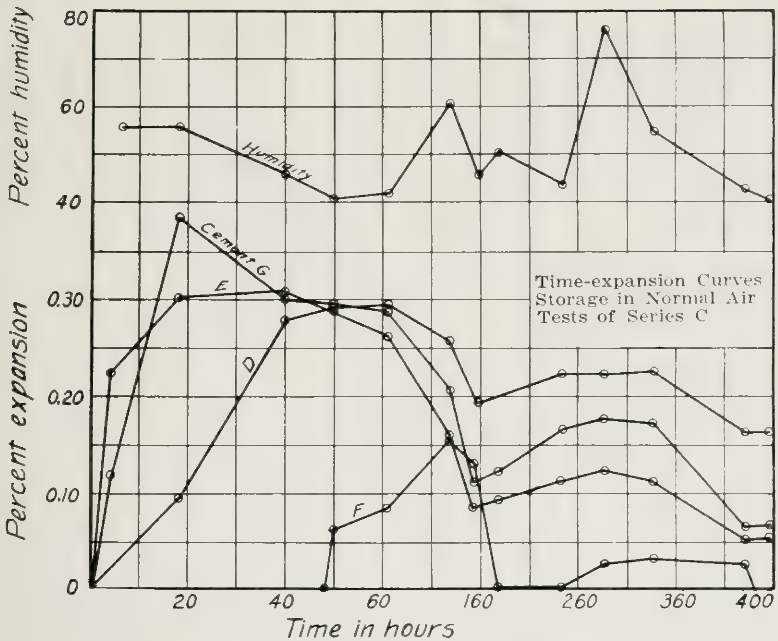

FIGURE 5.—Effect of Chemical Constitution of Cement

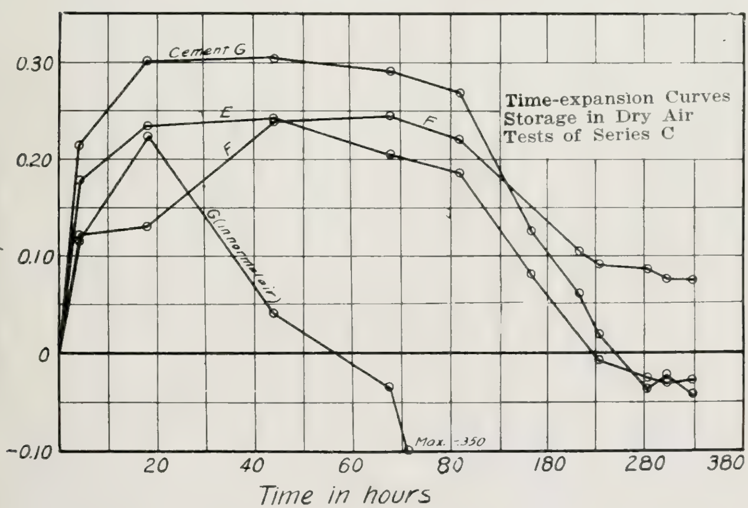

FIGURE 6.—Effect of Chemical Constitution of Cement

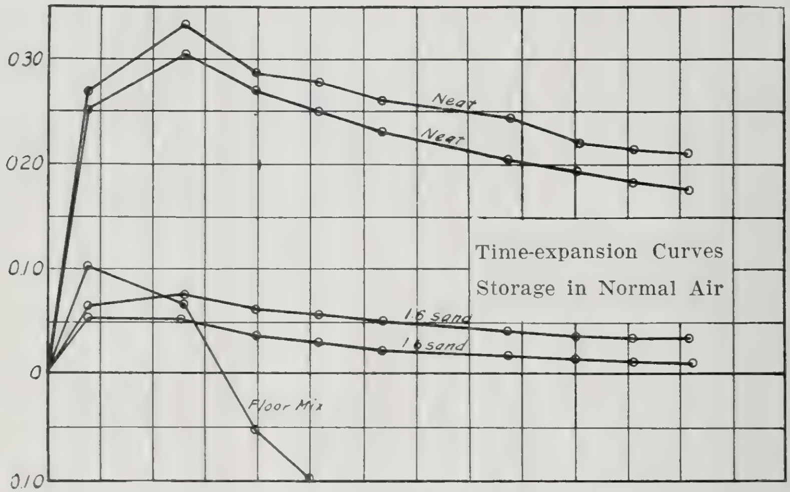

FIGURE 7.—Effect of aggregate on volume constancy

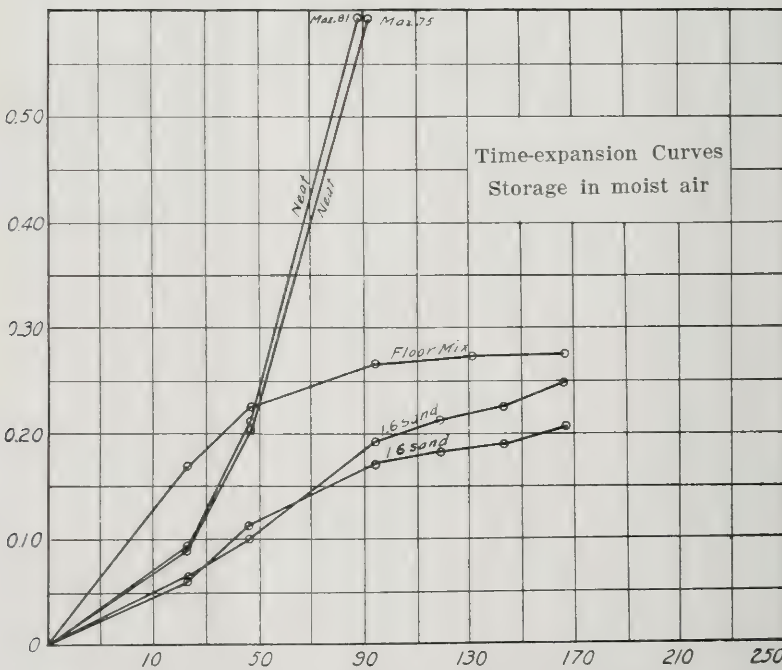

FIGURE 8.—Effect of aggregate on volume constancy

mens might be due in part to the elasticity of the aggregate, which would cause an expansion upon removal of the molds, and in part to expansion of the sawdust on account of moisture. Two tests were made to give information on this point. In the first test, two expansion specimens were made of a 1:6 sand mix, two of neat cement, and one of the standard floor mix. The same cement was used in all cases. These five specimens were tested together, and the time-expansion curves are shown in Fig. 7 and Fig. 8. From these curves, it appears evident that expansion is due to the action of the cement and $MgCl_2$, and not to the swelling of the sawdust.

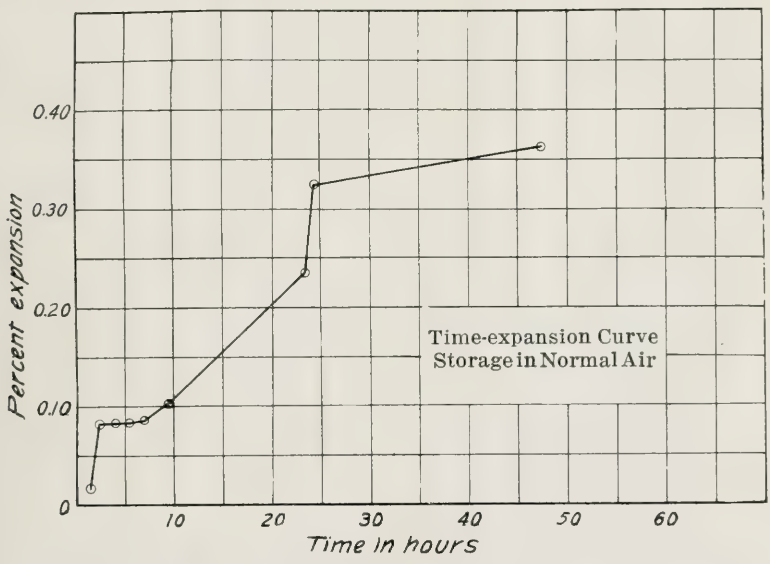

FIGURE 9.—Effect of Setting of Cement on Volume Constancy

In a second test, an expansion specimen was made of the standard floor mix, the molds were removed immediately after the material had been put in place, and the expansion of the specimen determined, from time to time, by means of a traveling microscope. The time-expansion curve so obtained is shown in Fig. 9. This curve seems to show that during the first two hours or so the specimen expanded due to the elasticity of the material, then remained practically constant in length for a time, and

then again expanded. It was decided to leave specimens in the molds not less than 8 hours, to eliminate the effect of expansion due to elastic recovery of the compressed aggregate and setting of the cement.

2—The Effect of Variation in the Amount of $MgCl_2$ Solution; Density Constant

Tests of Series D—The first series of tests made to determine the effect of variation in the amount of $MgCl_2$ solution consisted of tension and compression tests at 7 and 28 days on three different mixes: namely, a 1:3 sand mix, a 1:1 sawdust, and a 2:1 sawdust mix. The results of these tests are given in Table VII. Each tension value represents the average strength of four specimens; each compression value the average strength of three specimens.

TABLE VII

Effect of Variation In Amount of $MgCl_2$ Solution: Density Constant
Results of Tension and Compression Tests: Series D
The density of the $MgCl_2$ solution was 1.20
All specimens were made of the standard floor mix, with cement J

Mix	Amount Mg Cl_2 Solution c.c. per 100 gm. Solid Material	Ratio of Available Mg Cl_2 to Available MgO	Tensile Strength Lb. per Sq. In.			Compressive Strength Lb. per Sq. In.		
			7 Days	28 Days	Average both ages	7 Days	28 Days	Average both ages
1:3 Sand	12.6	.221	435	492	463	2820	2588	2704
	13.6	.238	491	538	515	3300	3566	3433
	14.6	.355	601	555	578	3600	3331	3466
	15.6	.273	493	492	493	3130	4266	3698
	16.6	.290	477	467	472	2980	4576	3778
1 Part cement to one part sawdust by weight.	88	.770	179	204	192	644	711	678
	98	.858	198	216	207	640	734	687
	108	.945	200	220	210	742	915	829
	118	1.033	177	210	193	619	812	716
2 Parts cement to one part sawdust by weight.	59	.387	536	583	559	1656	1776	1716
	64	.420	573	699	636	2522	2705	2614
	69	.453	502	630	566	2376	2570	2473
	74	.486	498	611	554	2369	2547	2458
	79	.518	456	487	472	2067	2212	2140

To facilitate a study of results, the diagrams of Fig. 10 have been drawn. These curves were obtained by plotting the average values in Table VII, against corresponding percentages of $MgCl_2$ solution.

It is of interest to note whether the strength of the mix was a maximum for that amount of the solution giving the theoretically ideal proportion of $MgCl_2$ to MgO. This amount has been computed, and is indicated on this and succeeding diagrams by the vertical line marked "I. P." The amount of solution required for normal consistency has been similarly indicated by a vertical line marked "N. C."

It will be seen that for all mixes except the 1:3 sand mix, the strength in both tension and compression is a maximum somewhere between the extreme proportions used, and for the sand mix this is true of the tensile strength. The maximum corresponds more nearly with the amount of solution required for normal consistency than with the amount required, according to the assumed formula, for ideal combination. The tensile strength and compressive strength seem to be similarly affected by the variation in amount of solution.

Tests of Series E—The second series of tests consisted of tension and hardness tests on specimens of a 1:6 sand mix and of the standard floor mix. The results are given in Table VIII and shown graphically in Fig. 11.

These results appear to disagree radically with those of Series D, and the results for the 1:6 sand and the floor mix are not consistent. There is, however, a very close agreement between the variation in strength and in hardness in the case of the floor mix.

Tests of Series F and G—In order to secure more consistent data on the point in question, two additional series of tests were performed, exactly alike, but made at different times. Tension, cross-breaking, and expansion tests were made, all strength specimens being tested at the age of 16 days. The floor mix only was used, the previous tests having shown that results obtained with a sand mix could not be regarded as comparable with those obtained with the floor mix.

The results of the tension and cross-breaking tests are given in Table IX and shown graphically in Fig. 12. It will be seen that

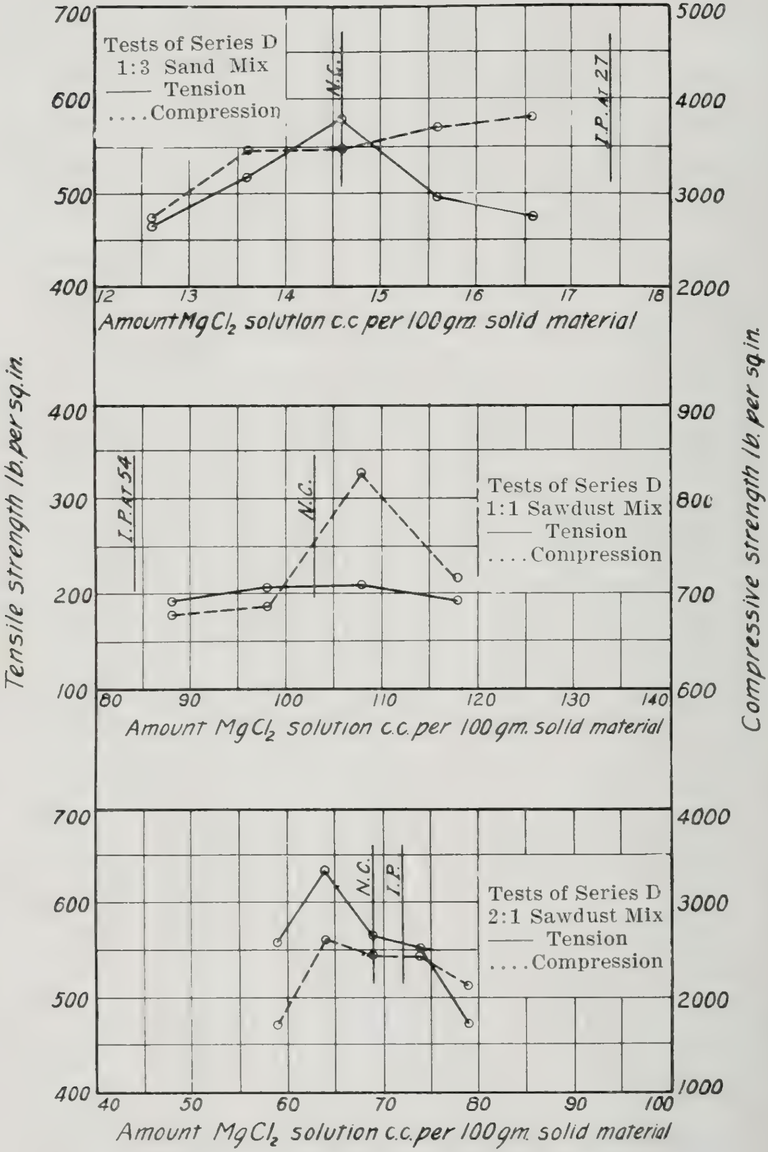

FIGURE 10.—Effect of Variation in Amount of $MgCl_2$ Solution

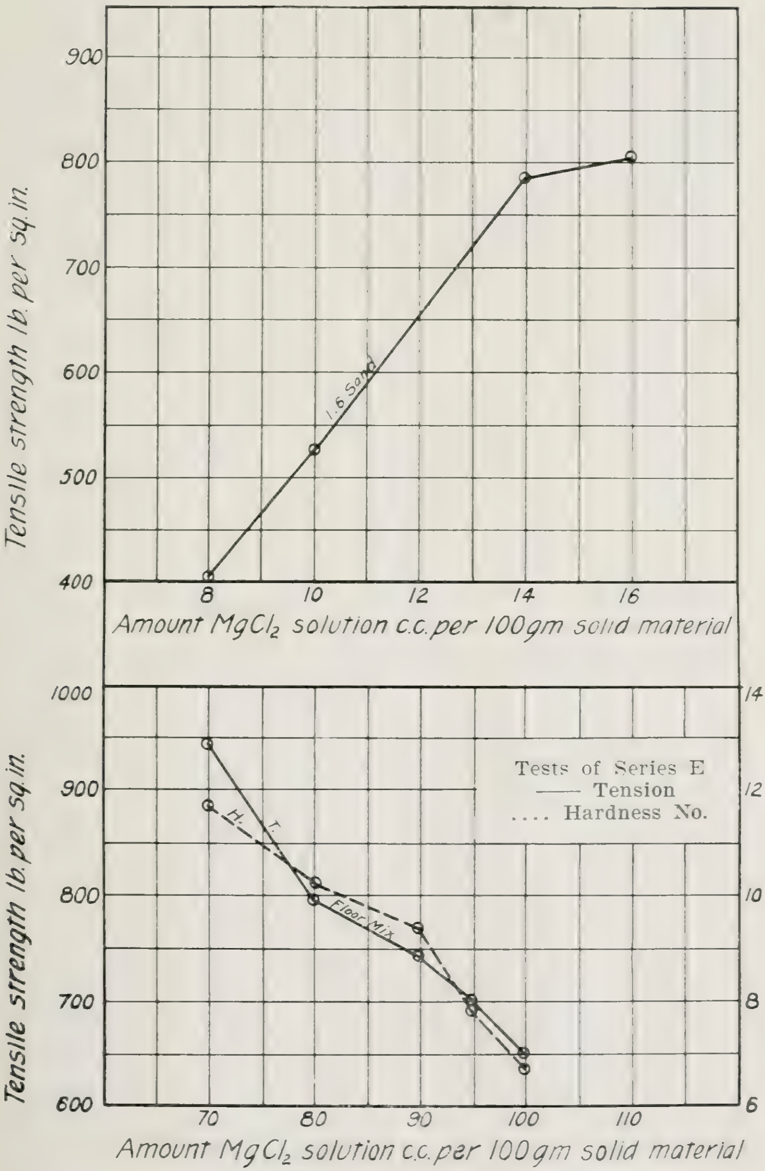

FIGURE 11.—Effect of Variation in Amount of $MgCl_2$ Solution

the results obtained are very consistent, the two series agreeing closely, and the variation in tensile strength, modulus of rupture, and modulus of elasticity being similar. The maximum strength and stiffness is seen to obtain for a slightly greater per cent of $MgCl_2$ solution than that required to give normal consistency.

TABLE VIII

EFFECT OF VARIATION IN AMOUNT OF $MgCl_2$ SOLUTION; DENSITY CONSTANT. RESULTS OF TENSION AND HARDNESS TESTS; SERIES E

The density of the $MgCl_2$ solution was 1.32 for the 1:6 sand mix, and 1.20 for the floor mix. Cement F was used

Mix	Amount $MgCl_2$ Solution c. c. per 100 gm. Solid Material	Ratio of Available $MgCl_2$ to Available MgO	Tensile Strength Lb. per Sq. In.			Brinell Hardness Number
			7 Days	28 Days	Average both ages	
1:6 Sand	8	.219	305	504	404	Test not made
	10	.274	419	634	526	
	14	.384	665	908	786	
	16	.436	665	948	806	
Standard Floor Mix	70	.364	935	952	943	11.75
	80	.416	728	865	796	10.23
	90	.468	726	756	741	9.40
	95	.494	583	817	700	7.80
	100	.520	601	698	650	6.70

The time-expansion curves for these two series of tests are shown in Fig. 13a and 13b and in Fig. 14a and 14b. The results for storage in dry air are rather irregular, but in general the least expansion and greatest subsequent shrinkage took place in the dryer mixes. For storage in moist air the results are more consistent; the expansion clearly varied inversely as the amount of solution.

Tests of Series H—A final series of tests was made, partly in order to check the results of the tests of Series F and G, and partly to determine the practicability of expansion tests made by measuring directly, with calipers, the change in dimensions

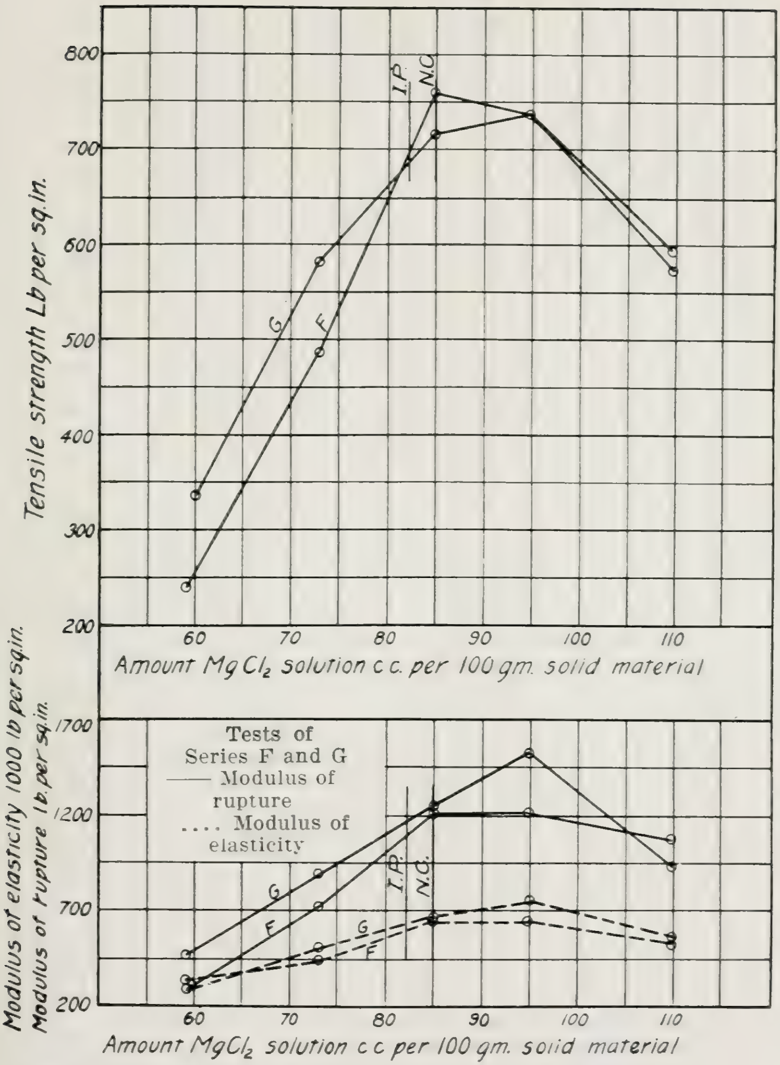

FIGURE 12.—Effect of Variation in Amount of $MgCl_2$ Solution

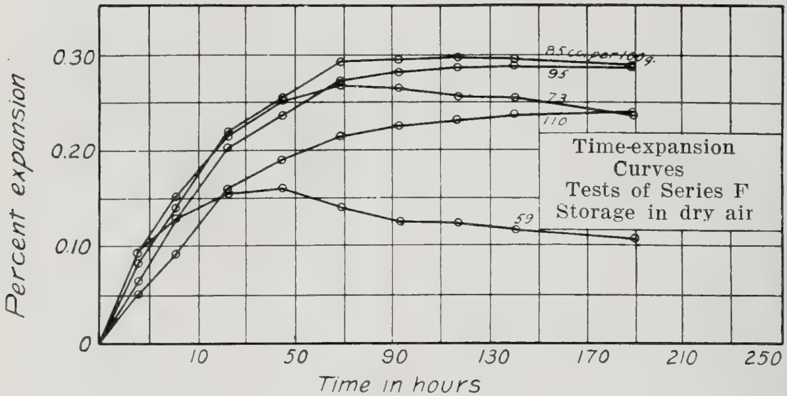

FIGURE 13a.—Effect of Variation in Amount of $MgCl_2$ Solution

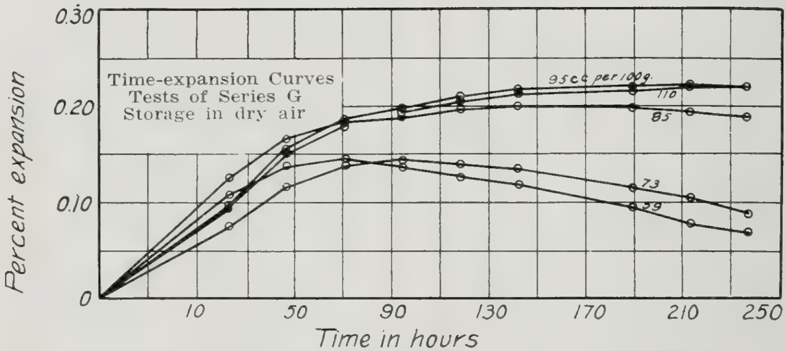

FIGURE 14a.—Effect of Variation in Amount of $MgCl_2$ Solution

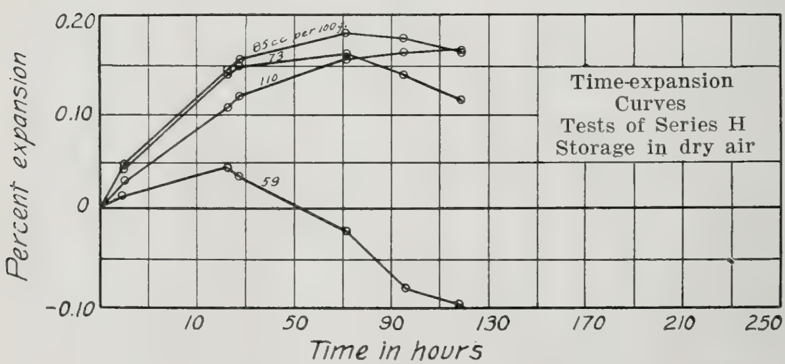

FIGURE 15a.—Effect of Variation in Amount of $MgCl_2$ Solution

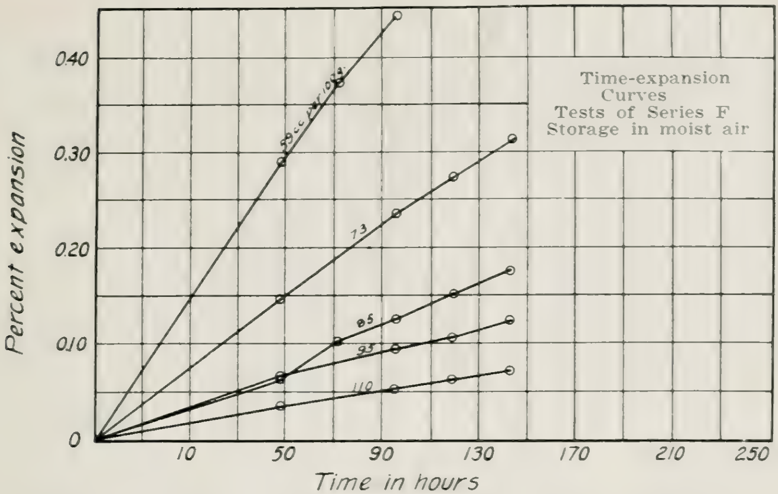

FIGURE 13b.—Effect of Variation in Amount of $MgCl_2$ Solution

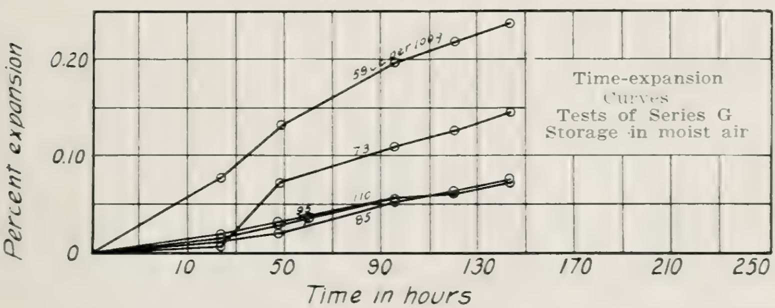

FIGURE 14b.—Effect of Variation in Amount of $MgCl_2$ Solution

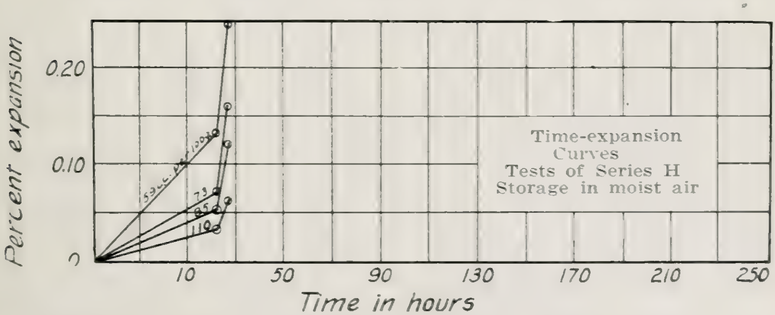

FIGURE 15b.—Effect of Variation in Amount of $MgCl_2$ Solution

TABLE IX

EFFECT OF VARIATION IN AMOUNT OF $MgCl_2$ SOLUTION; DENSITY CONSTANT RESULTS OF TENSION, CROSS BREAKING, EXPANSION, AND CHANGE IN WEIGHT TESTS; SERIES F AND G

The density of the $MgCl_2$ solution was 1.20
All specimens were made of the standard floor mix, with cement F

Series	Amount $Mg\ Cl_2$ Solution c.c. per 100g. Solid Material	Ratio Available $Mg\ Cl_2$ to Available MgO	Tensile Strength Lb. per Sq. In.	Modulus of Rupture Lb. per Sq. In.	Modulus of Elasticity Lb. per Sq. In.	Expansion and Change of Weight: Results in Dry Air: Change in length Per cent	Results in Dry Air: Change in weight Per cent	Results in Moist Air: Change in length Per cent	Results in Moist Air: Change in weight Per cent
F	59.0	.338	240	288	345000	Test not	made	Test not	made
	73.3	.420	487	720	432000				
	85.0	.486	762	1220	648000				
	95.0	.544	739	1220	648000				
	110.0	.629	572	1080	518000				
G	59.0	.338	337	468	281000	+0.056	− 4.5	+0.236	+2.9
	73.3	.420	582	900	508000	+0.076	− 7.8	+0.146	+2.9
	85 0	.486	718	1260	670000	+0.181	− 9.2	+0.079	+2.9
	95.0	.544	738	1530	756000	+0.215	−10.5	+0.077	+3.8
	110.0	.629	592	936	562000	+0.218	−11.3	+0.077	+5.0

of standard briquettes. It was thought important to devise some simple means of determining the constancy of volume of the material, so that an expansion test could be made with easily obtainable apparatus. Extreme conditions of storage were obtained by putting half the briquettes in a box containing water, and half in a box containing calcium chloride. Half of each group was measured and tested at four days, and half at eight days. The regular expansion tests were also made in this series, and the hardness test was included, but the cross-breaking test was omitted.

The results are given in Table X and shown graphically in Fig. 16. The time-expansion curves are shown in Fig. 15a and 15b. They show that in dry air the per cent expansion varied directly and in moist air inversely as the amount of $MgCl_2$ solu-

tion used. These results agree with those of Series F and Series G.

The results obtained from measurements made on briquettes are very irregular, and cannot be regarded as of value.

TABLE X

EFFECT OF VARIATION IN AMOUNT OF $MgCl_2$ SOLUTION: DENSITY CONSTAN RESULTS OF TENSION, HARDNESS AND EXPANSION TESTS ON BRIQUETTES: SERIES H

All specimens were made of the standard floor mix, with cement I

Amt. $MgCl_2$ solution c. c. per 100g. solid material	Ratio of available $MgCl_2$ to available MgO	Age days	Results in Dry Air			Results in Moist Air			Average for both ages and both methods of storage	
			Tensile ste'nth lb. per sq in.	Brinell hardness number	Per cent expansion	Tensile sten'th lb. per sq. in.	Brinell hardness number	Per cent expansion	Tensile sten'th lb. per sq. in.	Brinell hardness number
59	.352	4 8	443 562	6.7 6.6	+0.003 +0 012	312 370	5.4 5.1	+0.021 +0.071	422	6.0
73	.435	4 8	560 632	7.4 7.5	-0.003 -0.001	498 542	6.4 6.6	+0 046 +0.056	558	7.0
85	.506	4 8	502 570	6.7 7.3	+0.026 +0.056	470 508	7.0 6.9	+0.034 +0.059	512	7.0
110	.655	4 8	340 378	3.6 5.4	+0.051 +0 036	302 350	4.0 4.7	+0.041 +0.052	342	4.4

3—THE EFFECT OF VARIATION IN DENSITY OF $MgCl_2$ SOLUTION: AMOUNT CONSTANT

Tests of Series I—The first series of tests made to determine the effect of variation in density of solution consisted of compression tests on the standard floor mix, three different cements being used and three different densities of solution employed. The results are given in Table XI and shown graphically in Fig. 17. The strength is seen to increase with the density for all cases.

TABLE XI

Effect of Variation in Density of $MgCl_2$ Solution; Amount Constant
Results of Compression Tests; Series I

The amount of $MgCl_2$ solution was 85 c. c. per 100 g. of solid material
All specimens were made of the standard floor mix

Cement	Density of $MgCl_2$ Solution	Ratio of Available $MgCl_2$ to Available MgO	Compressive Strength Lb. per Sq. In.		
			7 Days	28 Days	Average
D	1.23	.875	1325	2630	1980
	1.20	.750	1135	2540	1840
	1.18	.670	1150	2380	1765
E	1.23	.563	3110	4890	4000
	1.20	.485	3120	5055	4090
	1.18	.432	2755	4400	3580
F	1.23	.567	2950	5410	4180
	1.20	.487	2855	5205	4030
	1.18	.435	2650	3725	3190

TABLE XII

Effect of Variation in Density of $MgCl_2$ Solution; Amount Constant
Results of Tension, Cross-Breaking, Expansion and Change in Weight Tests; Series J and K

The amount of $MgCl_2$ solution was 85 c.c. per 100 g. of solid material
All specimens were made of the standard floor mix with cement F

Series	Density of $MgCl_2$ Solution	Ratio of Available $MgCl_2$ to Available MgO	Tensile Strength Lb. per Sq. In.	Modulus of Rupture Lb. per Sq. In.	Modulus of Elasticity Lb. per Sq. In.	Expansion and Change of Weight			
						Results in Dry Air		*Results in Moist Air*	
						Per cent change in length	Per cent change in weight	Per cent change in length	Per cent change in weight
J	1.27	.677	839	1370	756000				
	1.23	.567	554	1570	800000				
	1.20	.487	496	432	518000	Test		Test	
	1.15	.357	225	216	260000	not		not	
	1.10	.115	0	0	....		made		made
K	1.27	.677	853	1300	756000	+0.210	— 1.17	+0.232	+5.80
	1.23	.567	590	1270	713000	+0.119	— 3.21	+0.134	+3.46
	1.20	.487	634	1370	670000	+0.079	— 7.60	+0.069	+2.53
	1.15	.357	442	830	497000	+0.079	—10.70	+0.031	+3 00
	1.10	.115	92	110	130000	+0.310	— 9.35	+0.063	+2.19

Tests of Series J and K—Two additional series of tests were made in which the standard floor mix only was used, and five densities of solution, ranging from 1.27 to 1.10 were employed. Tensile, cross-breaking and expansion tests were made, the tests of the two series being in all respects alike. The results of the strength tests are given in Table XII, and shown graphically in Fig. 18.

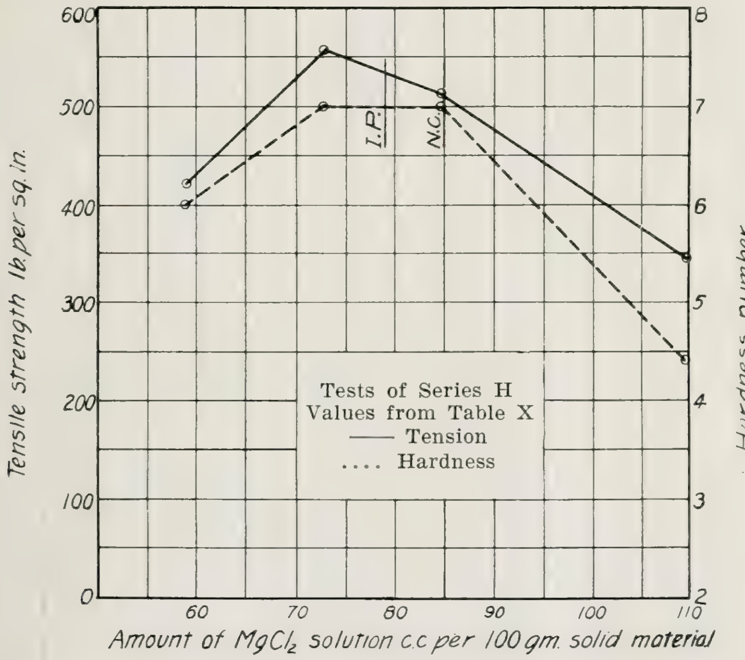

FIGURE 16.—Effect of Variation in Amount of $MgCl_2$ Solution

It is seen that the tensile strength increases with the density of the solution to a marked degree. The results of the cross-breaking tests are less consistent, but there is a marked decline in modulus of rupture and modulus of elasticity for densities below 1.20.

The time-expansion curves for these series are shown in Fig. 19a and 19b and in Fig. 20a and 20b. There is a marked inconsistency in the results here. In Series J the expansion in

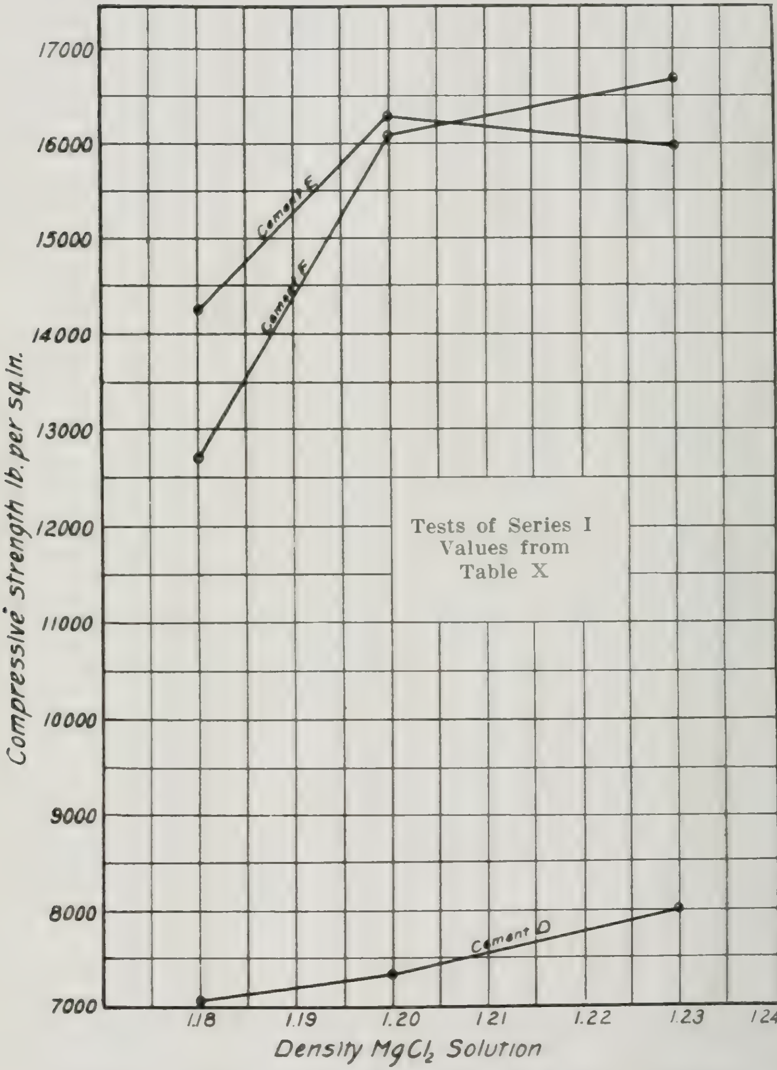

FIGURE 17.—Effect of Variation in Density of $MgCl_2$ Solution

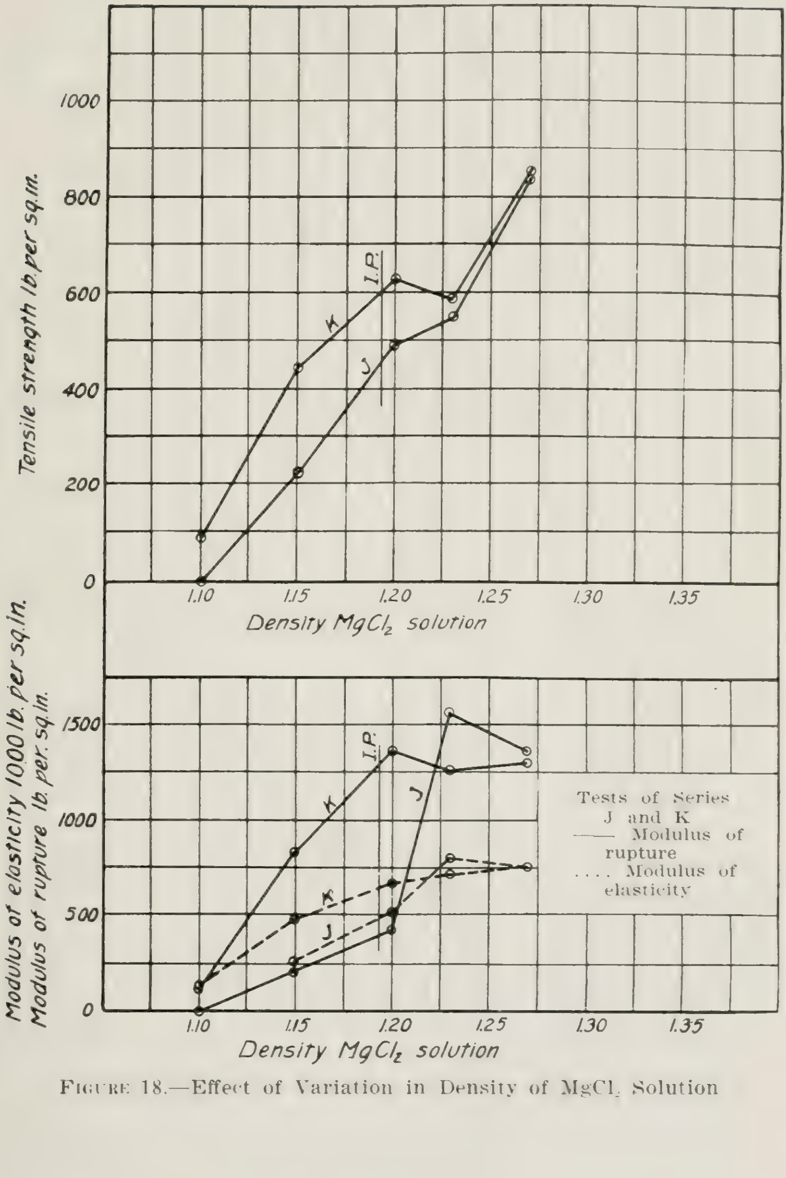

FIGURE 18.—Effect of Variation in Density of $MgCl_2$ Solution

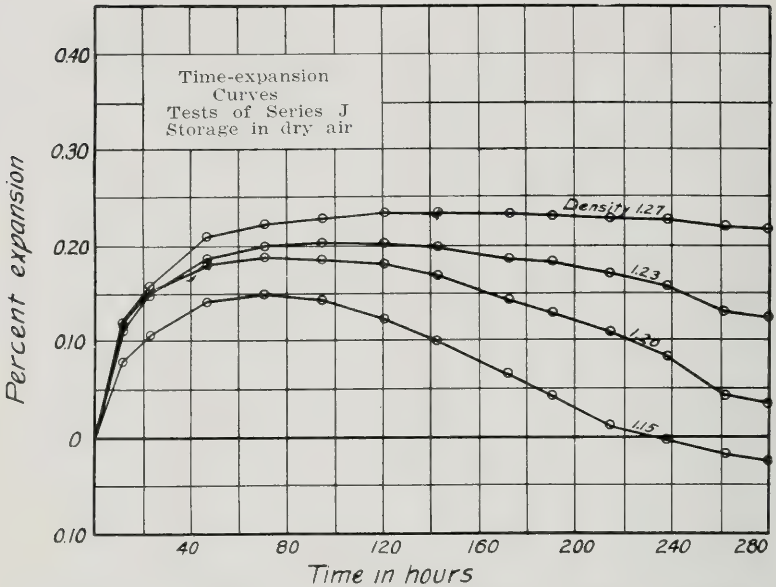

Figure 19a.—Effect of Variation in Density of $MgCl_2$ Solution

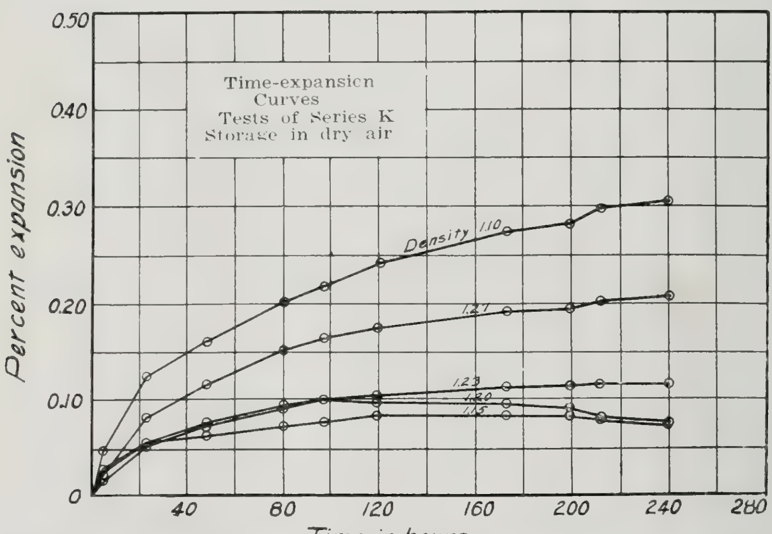

Figure 20a.—Effect of Variation in Density of $MgCl_2$ Solution

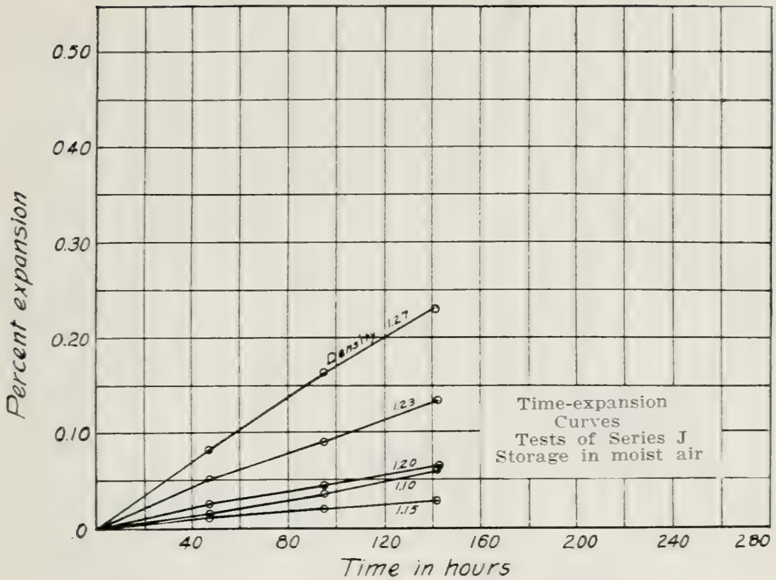

FIGURE 19b.—Effect of Variation in Density of $MgCl_2$ Solution

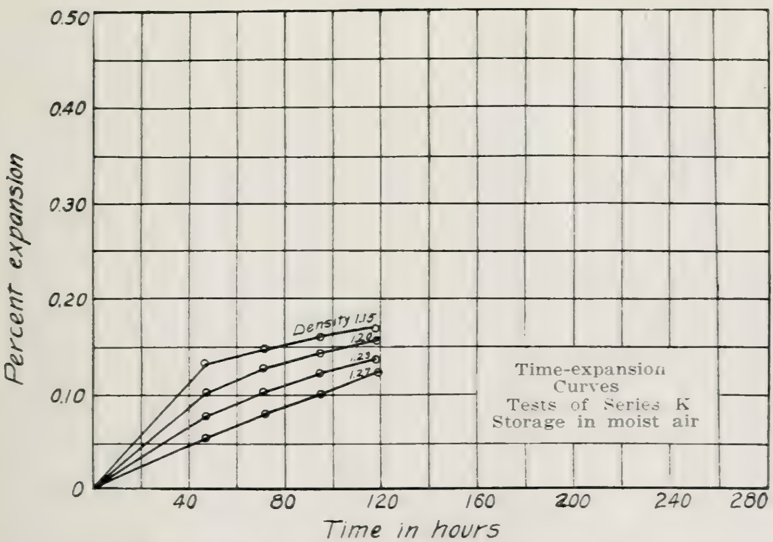

FIGURE 20b.—Effect of Variation in Density of $MgCl_2$ Solution

dry air is greatest for the mix containing the densest solution, while in moist air it is greatest for the mix containing the least dense solution. Series K shows similar results for storage in dry air, with the exception of the mix made with a solution of density 1.10. For storage in moist air, however, the results are directly contradictory to those secured before.

Tests of Series L— A fourth series of tests was made to settle the questions left in doubt by the former series. This series was similar to the Series J and K, except that in addition to the regular expansion test, measurements were made on briquettes as in Series H. The cross-breaking test was omitted and the hardness test included.

TABLE XIII

EFFECT OF VARIATION IN DENSITY OF $MgCl_2$ SOLUTION; AMOUNT CONSTANT

RESULTS OF TENSION, HARDNESS AND EXPANSION TESTS ON BRIQUETTES SERIES L

All specimens were made of the standard floor mix, with cement I

Density of $MgCl_2$ Solution	Ratio of Available $MgCl_2$ to Available MgO	Age Days	Results in Dry Air: Tensile Strength Lb. per Sq. In.	Results in Dry Air: Brinell Hardness No.	Results in Dry Air: Per cent Expansion	Results in Moist Air: Tensile Strength Lb. per Sq. In.	Results in Moist Air: Brinell Hardness No.	Results in Moist Air: Per cent Expansion	Averages for Both Ages and Both Methods of Storage: Tensile Strength Lb. per Sq. In.	Averages for Both Ages and Both Methods of Storage: Brinell Hardness No.
1.27	.705	4	552	7.7	0.33	485	6.6	0.44	516	7.0
		8	590	7.6	0.19	437	6.2	0.69		
1.25	.651	4	560	6.6	0.46	442	5.9	0.45	504	6.3
		8	532	6.6	0.52	482	6.2	0.68		
1.20	.507	4	492	5.4	0.26	460	5.3	0.38	479	5.4
		8	478	5.6	0.34	485	5.3	0.65		
1.15	.372	4	385	3.9	0.31	328	3.8	0.33	346	3.9
		8	362	3.9	0.12	308	4.1	0.42		
1.10	.120	4	70	2.8	0.45	55	2.4	0.69	65	2.5
		8	68	2.6	0.63	68	2.2	0.62		

The results of this series of tests are given in Table XIII and Fig. 21, and the time expansion curves are shown in Fig. 22a and 22b. It is seen that as before strength and hardness in-

crease with the density of the solution. The effect upon constancy of volume is not clearly defined, but in general it seems that the per cent change in length decreases with the density of the solution in a dry atmosphere, and increases in a moist atmosphere. The results obtained from measurements made on briquettes are irregular, and cannot be regarded as satisfactory.

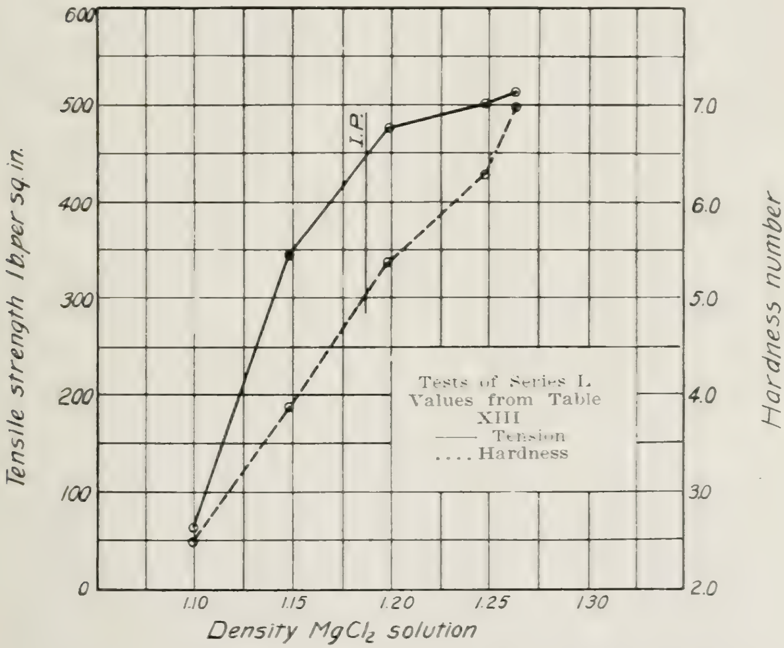

FIGURE 21.—Effect of Variation in Density of $MgCl_2$ Solution

Tests of Series M—It would seem from the tests described that no consistent relation could be established between volume constancy and density of $MgCl_2$ solution. One more series of tests was made, however, consisting of expansion tests only. The time-expansion curves obtained are shown in Fig. 23a and 23b. The results in dry air are not consistent with those obtained in Series M, but check those of Series J and K, showing an increased expansion with increased density of solution. In a wet atmosphere it is seen that during the first 40 hours or so of the

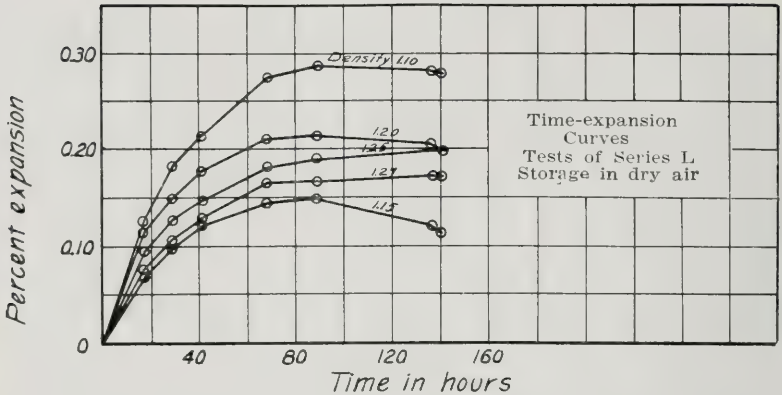

FIGURE 22a.—Effect of Variation in Density of $MgCl_2$ Solution

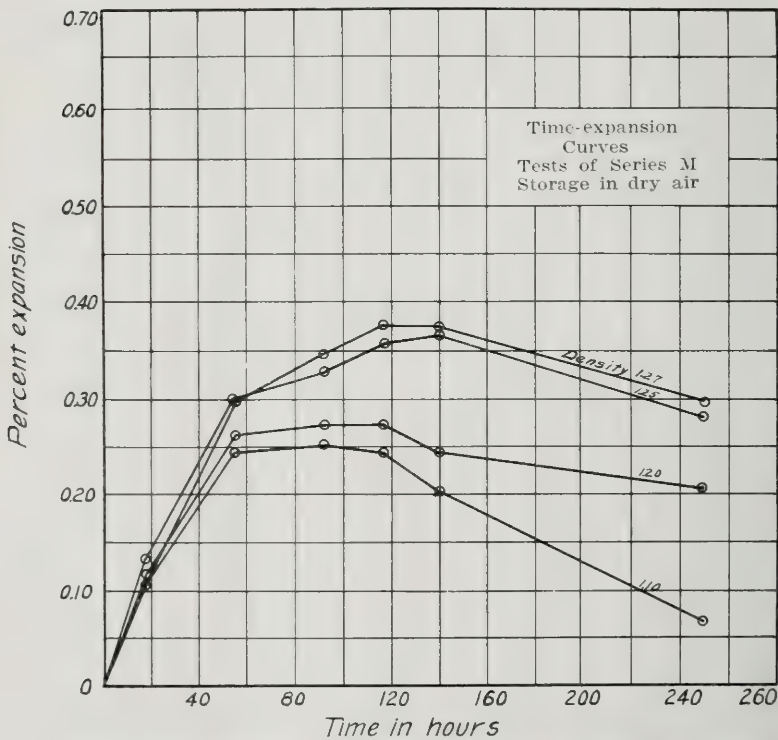

FIGURE 23a.—Effect of Variation in Density of $MgCl_2$ Solution

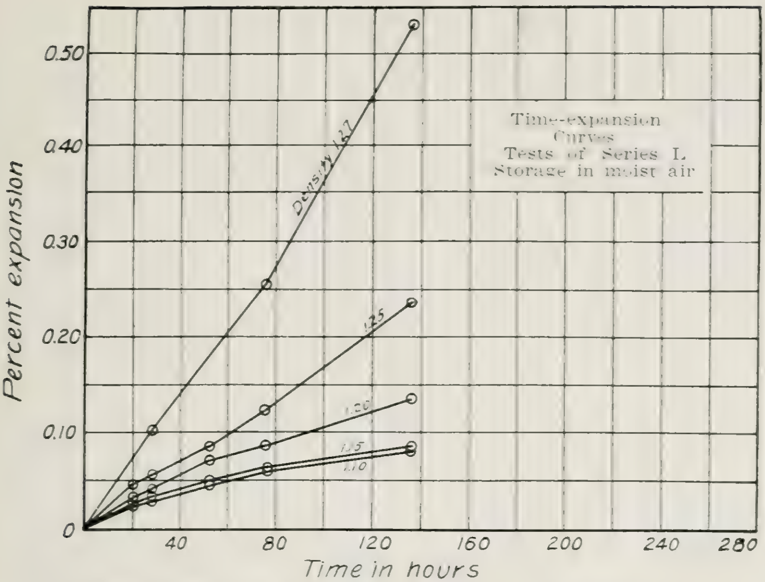

FIGURE 22b.—Effect of Variation in Density of $MgCl_2$ Solution

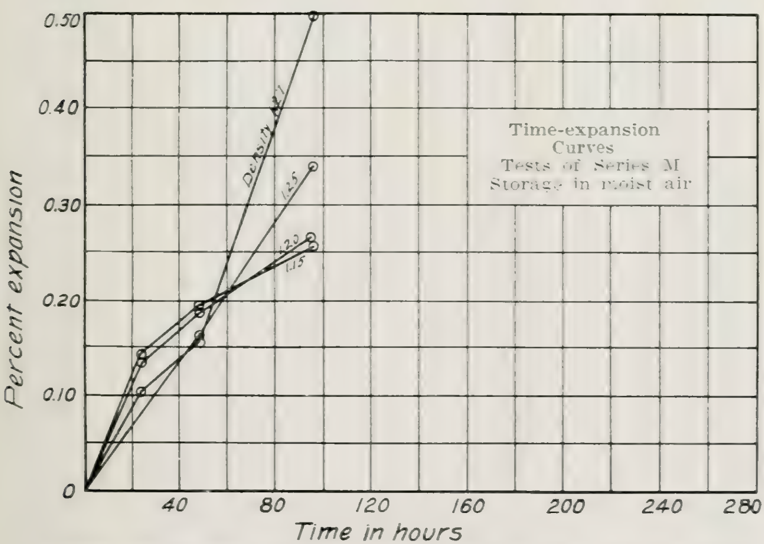

FIGURE 23b.—Effect of Variation in Density of $MgCl_2$ Solution

test the denser mixes expanded less than the others, but this order was reversed later, the final expansion being proportional to the density of the solution used.

If the curves for tests of Series J (Fig. 19b) are considered in the light of this fact, it will be seen that they converge with time, and it is quite possible that if the test had been extended over a longer period, the curves would have crossed and assumed an opposite order, as do those of Fig. 23b.

It was observed in the tests of Series I, J, K, L and M, that mixes made up with the denser solutions required less time to attain final set than did those made up with the weaker solutions.

4—The Effect of Variation in the Amount and Density of $MgCl_2$ Solution; Amount of $MgCl_2$ Constant

Tests of Series N and O—In two similar series of tests made to secure information on this point, tension, cross-breaking, and expansion tests were made on specimens of the standard floor

TABLE XIV

Effect of Variation in Amount and Density of $MgCl_2$ Solution: Amount of $MgCl_2$ Nearly Constant

Results of Tension, Cross-Breaking, Expansio and Change in Weight Tests: Series N and O

All specimens were made of the standard floor mix, with cement F

Series	Am'nt of Mg Cl2 Solution c.c. per 100 g. Solid Material	Density Mg Cl2 Solution	Ratio of Available Mg Cl2 to Available Mg O	Tensile St'gth Lb. per Sq. In.	Modulus of Rupture Lb. per Sq. In.	Modulus of Elasticity Lb. per Sq. In.	Expansion and Change of Weight			
							Results in Dry Air		Res'lts in Moist Air	
							Per Ct. Change in Length	Per Ct. Change in Weight	Per Ct. Change in Length	Per Ct. Change in Weight
N	39.2	1.31	.362	101	72	....				
	46.3	1.28	.383	176	432	260000	Test not			
	59.8	1.23	.399	330	324	195000			Test not	
	70.6	1.20	.408	372	847	454000		made		made
	80.8	1.18	.413	405	612	368000				
O	39.2	1.31	.362	93	126	....	+0.073	—6.0	+0.82	+5.5
	46.3	1.28	383	....	....	....	....	....	....	......
	59.8	1.23	.399	330	722	....	+0.071	—7.7	+0.433	+4 8
	70.6	1.20	.408	326	578	....	+0.086	—10.7	+0.286	+5.0
	80.8	1.18	.413	370	686	....	+0.071	—13.1	+0.197	+5.4

mix. It was planned to make the $MgCl_2$ content equal, in all specimens, to that in a mix of standard consistency made with a solution of density 1.20. Owing to a mistake in calculation, all the mixes were made up too dry, and not all had exactly the same amount of $MgCl_2$. The results of the strength tests are given in Table XIV, and it will be seen that the amount of $MgCl_2$ is very nearly the same for all specimens. The strength is seen to increase with the amount of $MgCl_2$ solution, but as there was in no case enough of the solution to produce normal consistency, the tests cannot be regarded as complete.

The time-expansion curves for this series are shown in Fig. 24a and 24b and Fig. 25a and 25b. The results are, on the whole, consistent, and show least volume change in both dry and moist air for low density and large amount of solution.

5—The Effect of Variation in Proportion of Cement to Aggregate

Tests of Series P—The tests of Series D, the results of which are given in Table VII, show that when the proportion of cement to sawdust was increased from 1:1 to 2:1, a marked increase in strength occurred. The results in question are not, however, strictly comparable, as the tests on the different mixes were not made at the same time. In order to secure results which would be comparable, a series of tests was made using the standard aggregates—asbestos and sawdust—in the usual proportion one to the other, and the standard amount and density of $MgCl_2$ solution, but varying the amount of cement from about half to about 150 per cent of that normally used.

The results of this series of tests are given in Table XV and in Fig. 26 and in Fig. 27a and 27b. As might be expected, strength and hardness are seen to increase with the proportion of cement used, but this increase becomes less marked as the mix is made richer. It seems that not much gain in either strength or hardness could be secured by using a mix richer in cement than the standard floor mix.

Study of the time-expansion curves shows that the amount of expansion, in both dry air and moist air, increases with the richness of the mix. The increase is relatively greater than the increase in tensile strength.

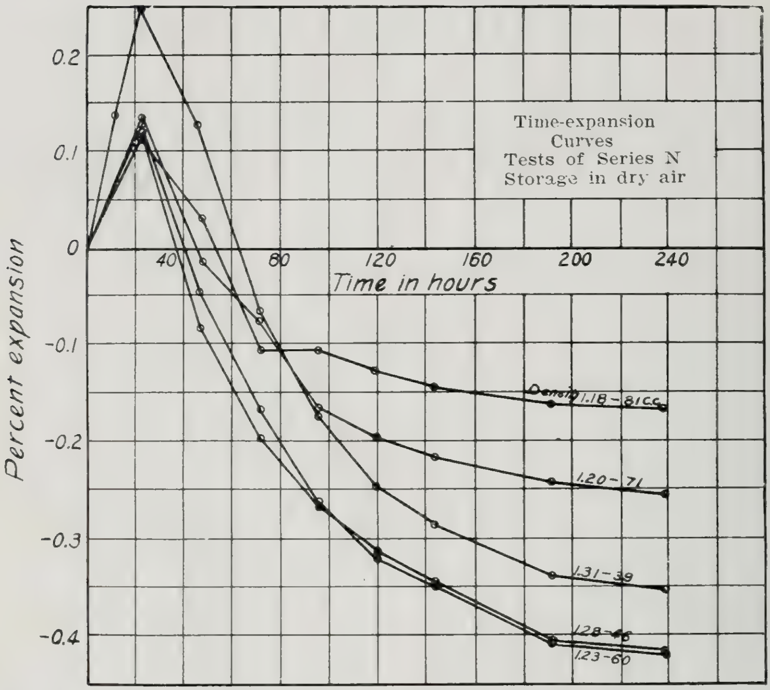

FIGURE 24a.—Effect of Variation in Amount and Density of $MgCl_2$ Solution

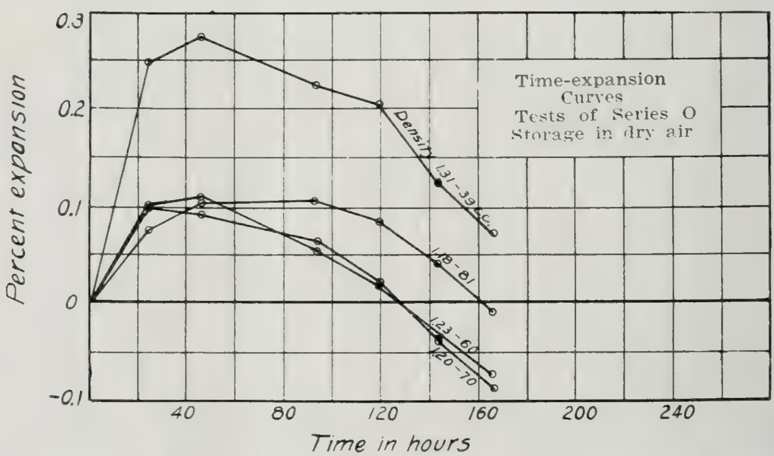

FIGURE 25a.—Effect of Variation in Amount and Density of $MgCl_2$ Solution

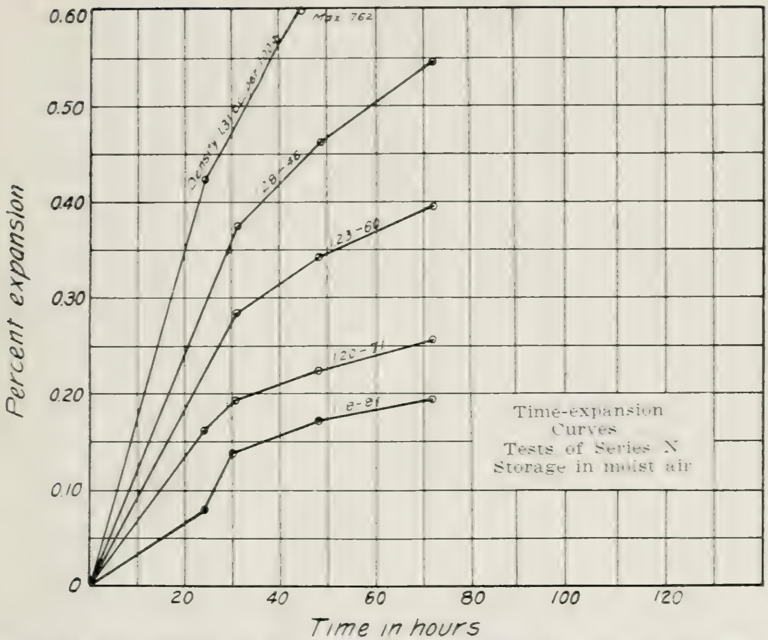

FIGURE 24b.—Effect of Variation in Amount and Density of $MgCl_2$ Solution

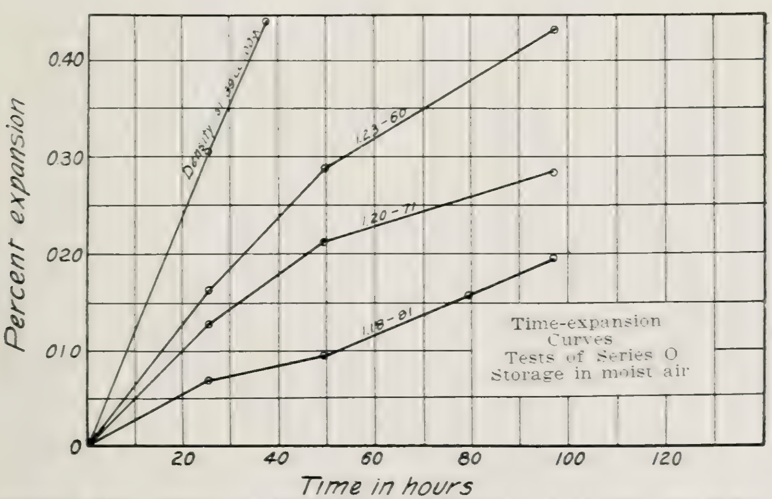

FIGURE 25b.—Effect of Variation in Amount and Density of $MgCl_2$ Solution

TABLE XV

EFFECT OF VARIATION IN PROPORTION OF CEMENT TO AGGREGATE
RESULTS OF TENSION, HARDNESS AND EXPANSION TESTS; SERIES P
All specimens were made of standard floor mix, with cement H

Proportions of Mix. Per Cent by Weight			Ratio of Cement to Aggregate per cent by Weight	Tensile Strength Lb. per Sq. In.	Brinell Hardness Number
Cement	Sawdust	Asbestos			
64	25	11	177	874	10.0
54	32	14	117	811	7.7
39	42	19	64	341	3.8

TABLE XVI

EFFECT OF VARIATION IN MOISTURE CONTENT OF AGGREGATE AND AMOUNT OF $MgCl_2$ SOLUTION; CONSISTENCY OF MIX UNIFORM
RESULTS OF TENSION AND COMPRESSION TESTS; SERIES Q
The density of the $MgCl_2$ solution was 1.20
The amount of $MgCl_2$ solution used was in all cases that required to give standard consistency

Per cent Moisture in Sawdust	Amount of $MgCl_2$ Solution c.c. per 100 gm. Solid Material	Tensile Strength Lb. per Sq. In.			Compressive Strength Lb. per Sq. In.		
		7 Days	28 Days	Average both ages	7 Days	28 Days	Average both ages
0	72.5	525	573	549	2340	2900	2620
8.1	70.0	502	630	566	2376	2570	2473
20.0	66.3	481	637	559	2102	2548	2325
40.0	60.2	419	567	493	1863	2304	2084
60.0	54.0	401	467	434	1340	1658	1499

6—Effect of Variation in Moisture Content of Aggregate

Tests of Series Q—It seems apparent that a portion of the solution, as ordinarily used in the floor mix, must be absorbed by the sawdust, the $MgCl_2$ in this part being practically wasted.

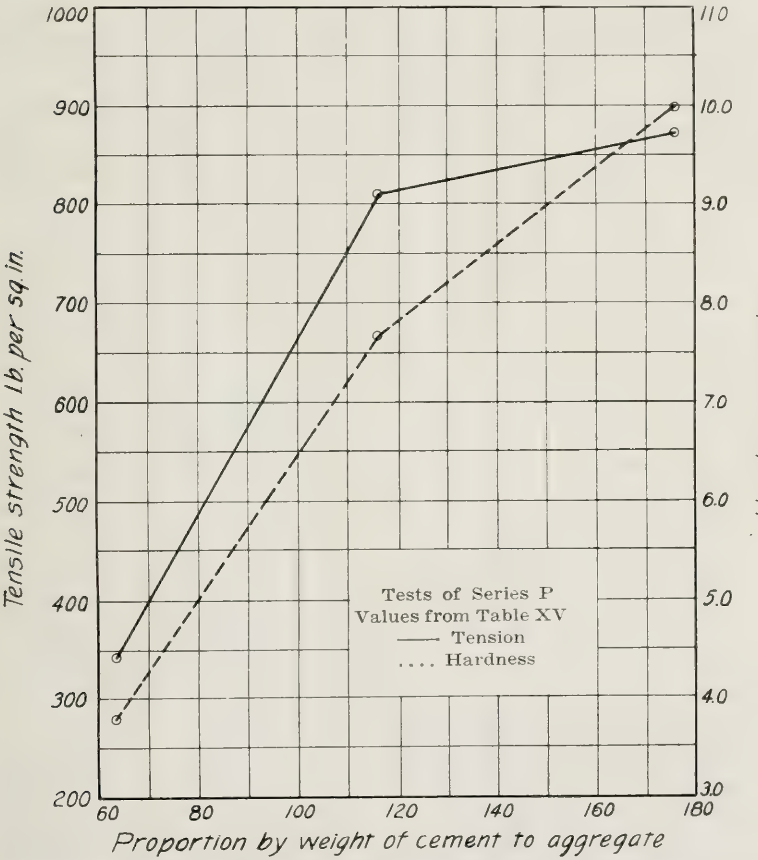

Figure 26.—Effect of Variation in Proportion of Cement to Aggregate

It was thought that by wetting the aggregate prior to mixing, normal consistency might be obtained with a relatively small amount of $MgCl_2$ without undue sacrifice of strength.

A series of tests, consisting of tension and compression tests on a 2:1 sawdust mix, was made to investigate this point. Wa-

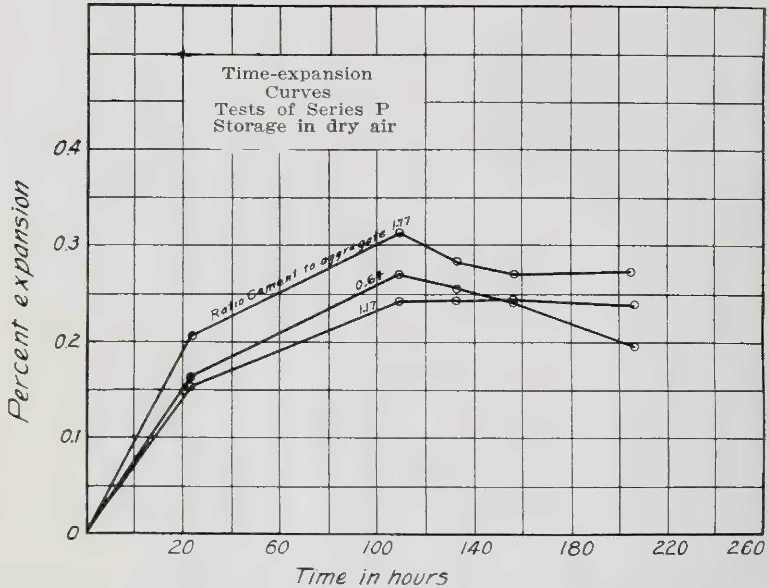

FIGURE 27a.—Effect of Variation in Proportion of Cement to Aggregate

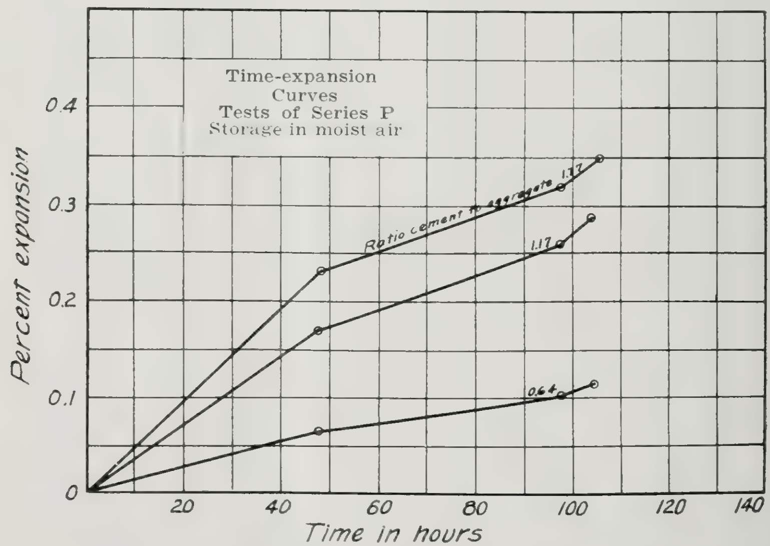

FIGURE 27b.—Effect of Variation in Proportion of Cement to Aggregate

ter, in amounts ranging from 0 to 60 per cent by weight of the aggregate, was added prior to mixing. Sufficient $MgCl_2$ solution, at a density of 1.20, was then used to give normal consistency. The results of these tests are given in Table XVI. Both the tensile and compressive strength are seen to decrease as the amount of $MgCl_2$ solution decreases, and a comparison of these results with those given in Table VII shows that this decrease is fully as great when the sawdust is dampened as when it is dry. It does not appear, therefore, that any advantage, as far as securing equal strength with less $MgCl_2$ is concerned, attaches to moistening the aggregate. It would be possible, however, to secure a normal consistency with less of the solution in this way, and the decrease in strength might not be serious in certain instances.

DISCUSSION OF THE PHYSICAL PROPERTIES OF THE FLOORING COMPOUND

From the results of the tests already described, average values may be obtained for the various physical properties of the mixes tested. The results of different tests of course differ considerably, but it is possible to determine more or less representative values, and to arrive at certain conclusions as to what may be expected under ordinary conditions. A comparison of the physical properties of the material under discussion with those of other materials used for the same purposes will also be of interest.

Tensile Strength—Of 108 neat cement briquettes mixed to standard consistency with a solution of density 1.20, stored in normal air and tested at ages of from seven to twenty-eight days, the average strength was 658 lb. per sq. in. The minimum average for any group of four similar briquettes was 0, and the maximum 1163 lb. per sq. in. Only four groups of four specimens averaged below 300. The average per cent departure from the mean, for 27 groups of four each was 16.4, and for two such groups the per cent departure was more than 40.

Of 108 1:3 sand briquettes similarly made, stored, and tested,

the average strength was 624 lb. per sq. in. The maximum average for any group of four was 1030, and the minimum 229 lb. per sq. in. Only three groups of the 27 averaged below 300. The average per cent departure from the mean was 7.2, and only three groups showed a per cent departure of more than 15.

Of 84 briquettes of the standard floor mix, made with 85 c. c. of $MgCl_2$ solution of density 1.20 per 100 g. of solid material and stored in normal air, the average strength was 593 lb. per sq. in. The maximum was 934, and the minimum 282 lb. per sq. in. Only four of the 21 groups tested averaged less than 400. The average per cent departure from the mean was 4.2, and only three groups showed a departure greater than 6 per cent.

Compressive Strength—The average compressive strength of cubes made of the standard floor mix, with 85 c. c. of solution of density 1.20 per 100 g. of solid material and stored in normal air, was 2340 lb. per sq. in. at 7 days and 4000 lb. per sq. in. at 28 days. The maximum was 3100, and the minimum 1100 lb. per sq. in. at 7 days, while the maximum and minimum were respectively 5400 and 2400 lb. per sq. in. at 28 days.

Cross-Breaking Strength and Modulus of Elasticity—The average modulus of rupture of specimens of the standard floor mix, tested at 7 days, was 1200 lb. per sq. in. The minimum value obtained was 432, and the maximum 1370 lb. per sq. in. The average value of the modulus of elasticity was about 650,000 lb. per sq. in. Failure generally took place at a center deflection of from 0.04 to 0.06 in.

Hardness—The average hardness number obtained for the standard floor mix was 7.3. As has been explained, this hardness number represents the pressure, in kilograms per square millimeter, required to produce permanent indentation. Expressed in lb. per sq. in., this quantity would be 10400. As would be expected, this is far in excess of the compressive strength of the material, as determined by tests on cubes. It may be said to represent that unit pressure which, extended over even a very small portion of the floor surface, would cause crushing.

Volume Constancy—The general behavior of all the expansion specimens made of the floor mix was essentially the same. On being placed in a dry atmosphere immediately after removal from molds, the specimens expanded rapidly for a time,

then less rapidly, and finally began to shrink or to maintain a nearly constant length. On being then placed in a moist atmosphere, the specimens began to expand, and this expansion continued as long as the specimens were permitted to remain under such conditions of storage,—in one instance, for as much as six weeks. The amount of shrinkage and expansion varied considerably in the different tests, but for the standard floor mix, gauged to normal consistency with a solution of density 1.20, the average expansion in dry air was 0.22 per cent. The maximum expansion observed in dry air was 0.30 per cent, the minimum 0.10 per cent, and the maximum shrinkage 0.23 per cent. The test in which expansion was measured from the time the specimen was made, showed that expansion took place during setting, and this seems to be true of the material in general. The effect of change in volume during setting would not, however, be so great as the effect of change in volume after setting, as the material, while yet plastic, could accommodate itself to such changes without sustaining high stresses.

Of course the behavior of the expansion specimens used in the tests described cannot be regarded as fairly representative of the behavior of a similar mix when laid as a floor. The fact that the floor layer is protected from the air on all sides except the top, which is sealed to some extent by troweling, together with adhesion to the floor base, would probably reduce the tendency either to expand or contract. The experimental results should be regarded as indicative of the maximum, rather than the average volume change which would tend to occur in a floor.

Eventually many of the expansion specimens became curved, in some cases departing at mid-length as much as 0.13 in. from the original axis. In all cases the bottom side of the specimen, as molded, lay on the concave side of the curve. There seemed to be no consistent relation between the degree of this curvature and the expansion or shrinkage observed during the test.

Coefficient of Thermal Expansion—The results of three tests on the standard floor mix gave, as the value for coefficient of thermal expansion, 0.0000126 per degree Fahrenheit. The specimens tested,—which were standard expansion specimens,—were heated in an oil bath, and it is possible that the oil may have affected the results. No change in length was observed while the

specimens remained immersed in oil at constant temperature, however.

The coefficient obtained is considerably higher than that of concrete (0.0000060), or of pine (0.0000030 along the grain). The difference might lead to difficulty in securing perfect adhesion between the composition and a base of either wood or concrete, in cases where the floor was subject to wide variations of temperature. Thus, taking the case of composition laid on a pine base, and assuming a change in temperature of 70 degrees F., the difference in expansion or shrinkage would cause a stress in the flooring of about 430 lb. per sq. in. If the base were of concrete, the stress corresponding to this range in temperature would probably amount to about 300 lb. per sq. in. This, however, would not cause cracking in a mix of normal strength.

TABLE XVII

PHYSICAL PROPERTIES OF VARIOUS FLOORING MATERIALS

MATERIAL	TENSILE STRENGTH LB. PER SQ. IN.	COMPRESSIVE STRENGTH LB. PER SQ. IN.		HARDNESS NO. (ACROSS GRAIN FOR TIMBER)	MODULUS OF RUPTURE LB. PER SQ. IN.	MODULUS OF ELASTICITY LB. PER SQ. IN.	WEIGHT LB. PER CU. FT.	COEF. OF THERMAL EXPANSION PER °F.	
		Along grain	Across grain					Along grain	Across grain
Yellow pine	....	8000	1400	6.5	14,000	2,000,000	38	.0000030	.000019
Hard maple	....	9000	1700	6.7	15,000	1,800,000	43	.0000027	.000030
White oak	....	8000	1600	6.9	14,000	1,600,000	50	.0000036	.00002
Neat Portland Cement	750	5000		40.0	930	5,000,000	160	.0000060	
Paving brick	....	15000		57.0	2,500	7,000,000	145	.0000033	
Marble	....	12700		....	1,400	7,000,000	165	.00000085	
Composition Flooring	....	....		....	....	....	....	....	
Standard Floor. Mix.	590	4000		73.0	1,200	650,000	74	.0000126	
Commercial Sample No. 1.	....	....		45.0	....	....	....	....	
No 2.	....	....		17.0	....	....	....	....	
No 3.	....	....		56.0	....	....	....	....	
No. 4.	....	....		17.0	....	....	....	....	
No. 5.	....	....		43.0	....	....	....	....	
No. 6.	....	....		71.0	....	....	....	....	

Comparison of Composition Flooring with Other Flooring Materials—In order to facilitate a comparison of the physical properties of composition flooring with those of other materials used for the same purpose, the data of Table XVII are pre-

sented. In addition to the values for the standard floor mix, hardness numbers are given for several of the patented compositions as made up by different manufacturers. The tests from which these values are taken were made on samples of the finished flooring, as sent out by the manufacturers.

THE RELATIVE VALUE OF DIFFERENT TESTS—SPECIFICATIONS

In discussing the relative value of the different tests with regard to their inclusion in specifications, account must be taken not only of the information yielded, but also of the ease and convenience with which the test can be made. Thus, considered wholly on the basis of value of results, the tensile test is perhaps not so important as the expansion or the compression test. The fact that it can be easily and quickly made with apparatus generally available would, however, make it particularly valuable if it could be shown to provide significant data. It has been seen that the tensile strength is, in general, proportional to the hardness, compressive strength, cross-breaking strength, and modulus of elasticity, and that it is similarly affected by the same factors and conditions. It seems, therefore, that the tensile strength could be accepted as an index to the general strength and soundness of the material, and that it could be relied upon to detect marked inferiority in a given cement or composition. The tensile strength, however, gives no information whatever as to the tendency of the material to shrink or expand. Only tests in which the expansion or shrinkage is actually measured can be relied upon to give such information. Study of the results given in Tables XII and XIV shows that there is no consistent relation, at least no quantitative relation, between change in weight and change in volume. The pat test, when made on the floor mix, seems to give purely negative results, as in no instance was there any indication of cracking or disintegration. On the other hand, pat tests on neat cement showed marked difference in behavior for different cements but there was no connection between the behavior of such pats and the be-

havior of expansion specimens made of the standard floor mix in which the same cements were used.

The main drawback of the expansion tests as made is the difficulty of general application, due to the special apparatus required. The attempt to attain the same ends by calipering the briquettes was made in the hope of devising a simple form of expansion test that could be made with apparatus ordinarily available. The results secured were by no means satisfactory, and it is believed that this was due, in part, to the limitations in accuracy imposed by the instruments used, and in part to the fact that satisfactory surfaces between which to measure were not established. In the first series of such tests, tacks were set in the edges of the briquettes, so that the heads would provide points between which measurements might be made. Considerable difficulty was experienced in getting the tack heads in parallel planes and exactly opposite each other, and it was found that when this was not done, consistent measurements were not possible. In the second series of tests of this sort, measurements were made directly on the briquette, the mean of three measurements being taken. Better results were secured in this way, but the small transverse dimension of the briquette, $1\frac{5}{8}$ in., makes the relative accuracy of such measurements too low for the purpose. It would seem, from a consideration of the tensile strength, modulus of elasticity, and average expansion of the floor mix, that to be of value any expansion test should determine the amount of expansion to within about 0.005 per cent; this would correspond to a difference in stress in the material of about 30 lb. per sq. in. If the measurements were made on briquettes, this would mean that they should be made to the nearest 0.00008 in., and this cannot be done with the ordinary micrometer calipers reading to thousandths directly and ten-thousandths by estimation. The measurements made on the regular expansion specimens were, it is believed, amply accurate. Indeed, in such a test a length of specimen of 10 in. would be sufficient, and of course more convenient. An apparatus such as that used can be constructed at a cost of about $35, and by reducing the size of the box and the number of micrometer screws, this could probably be reduced one-half.

On the whole, it is believed that a combination of tensile tests

and expansion tests, somewhat along the lines of those employed in this investigation, would be found valuable as a means for determining the fitness of a given cement or of a given composition for use as flooring. Such tests should be made on the composition mixed exactly as for use in the floor, and not on the neat cement, nor on an arbitrary standard mix such as a 1:3 sand mortar. Before specifications could be written stating the requirements as to the results of such tests, it would be necessary to study the behavior of the material in floors in conjunction with the results of laboratory tests. If it could be shown that a mix that expanded a certain amount and shrunk a certain amount in the expansion test, and that had a certain tensile strength, gave satisfactory results in use, it would be known that such test values were not an indication of inferiority. If, on the other hand, higher values for the per cent expansion and shrinkage, and lower values for the tensile strength were obtained on a mix that failed in use, such test values could be regarded as an indication of poor quality. In this way, and in this way only, would it be possible to determine the limiting values that should be specified. In order to so correlate laboratory tests and practical application, cooperation between the investigating laboratories and manufacturers is, of course, essential. It was hoped that such cooperation might be arranged for in connection with this investigation, and no difficulty was experienced in finding manufacturers willing to lend assistance. The condition in the composition floor industry temporarily brought about by the cutting off of the foreign supply of materials, however, prevented this being done. Such cooperation should certainly be sought in connection with any further study of the subject.

SUMMARY

The results of the investigation may be regarded as giving information on (I) methods of testing and (II) the physical properties of the material tested and the effect on these properties of certain factors. The conclusions that appear warranted may be summarized under these two general heads as follows:

Methods of Testing

1. All tests should be made on specimens composed of the mix to be used in the floor. Results of tests made on neat cement, or on sand mortar, are not an index to the behavior and properties of a compound made with the same cement and the aggregates commonly employed in practice.

2. All specimens the results of tests on which are to be directly compared, should be made at the same time, by the same operator, and stored and tested under identical conditions.

3. The method of mixing and molding specified with reference to the standard test of Portland cement may be used, except that mixing should be done in a watertight and nonabsorbent vessel, in order to prevent any loss of the solution.

4. Specimens should be allowed to remain in the molds for at least eight hours.

5. Specimens may be stored exposed to ordinary room atmosphere, but should not be stored where the relative humidity is extremely high, say above 80 per cent.

6. The results of tension tests, made on standard briquettes in the manner specified for Portland cement, may be regarded as an index to the compressive strength, flexural strength, modulus of elasticity, and hardness of the material. These properties are in general proportional each to the others, and are similarly affected by any given factor.

7. The tendency to expand or to shrink can be determined only by actual measurement. Variation in weight is not proportional to variation in volume, and the pat test, whether made on neat cement or on the floor mix, appears to be of little or no value.

8. The physical tests which appear to be most practicable and to yield information of most value, are the tension and expansion tests.

9. Before limiting values for the results of these or other tests can be specified, it will be necessary to effect a correlation between the behavior of the material in service and in laboratory tests.

Nature of the Material and Effect of Various Factors

1. The standard floor mix, gauged with a magnesium chloride solution having a density of 1.20, in the proportion of 85 c. c. of solution to 100 g. of solid material, was found to have:

An average tensile strength of 593 lbs. per sq. in.

An average compressive strength of 2340 lbs. per sq. in. at 7 days, and of 4000 lbs. per sq. in. at 28 days.

An average modulus of rupture of 1200 lbs. per sq. in.

An average modulus of elasticity of 650,000 lbs. per sq. in.

An average Brinell hardness number of 7.3.

A coefficient of expansion of 0.0000126 per degree F.

2. Specimens made of this mix were found, in general, to increase in strength with time. The increase in compressive strength was very much more marked than the increase in tensile strength; in some instances, a decrease in tensile strength was observed.

3. Specimens made of this mix were found to expand during setting, and for some time afterward. In a dry atmosphere the maximum expansion amounted to from 0.10 to 0.30 per cent, and occurred in from 70 to 140 hours. After that time the specimens began to shrink, in some cases attaining a length less than the original length. In a moist atmosphere the specimens expanded continuously for as long a time as observed. The rate of expansion varied greatly, but in general decreased after a few days.

4. In general, the weight of specimens decreased during storage in a dry atmosphere, and increased during storage in a moist atmosphere.

5. In general, it was found that the strongest mixes were those made with the cements having a high magnesium oxide content

and a low lime content. It was also found that the addition of lime to the cement produced a marked decrease in tensile strength. No consistent relation was found to obtain between the strength and the amount of any other chemical constituent of the cements.

6. The tendency to shrink or expand was not found to bear any direct relation to the chemical constitution of the cement.

7. Maximum strength and hardness were obtained with about that amount of magnesium chloride solution necessary to give a good working consistency,—that is, about 85 c. c. of solution per 100 g. of solid material. The use of much more or much less resulted in a decrease of strength and hardness. This amount of solution corresponds fairly closely to the amount required, according to the assumed formula for the composition of magnesia cement, for perfect combination between the magnesium oxide and the magnesium chloride.

8. In a dry atmosphere the per cent expansion of the specimens was found to increase with the amount of solution used; in a moist atmosphere the per cent expansion was found to decrease with the amount of the solution used.

9. In dry air the per cent loss in weight of the specimen was found to increase with the amount of solution used; in moist air the per cent gain in weight was found to increase with the amount of solution used.

10. The strength, stiffness, and hardness were found to increase with the density of the magnesium chloride solution used. Mixes made with a solution having a density less than 1.15 were found to have very little strength. A density of 1.15 would seem to be the least that should be employed in any case.

11. In a dry atmosphere the per cent expansion of the specimens was found to increase with the density of the solution; in a moist atmosphere, it was found that the per cent expansion at first varied inversely, and later directly, as the density of the solution. The relation between per cent expansion and density of solution did not, however, appear to be very clearly defined.

12. In dry air the per cent loss in weight of the specimens was found to vary inversely as the density of the magnesium chloride solution used. In moist air the per cent gain in weight was found to vary directly as the density of the solution.

13. The results of tests made on mixes in which different amounts and densities of solution had been used, but in which the amount of magnesium chloride was practically the same, showed that strength and hardness increased with the amount of solution, so long as this amount was less than that required to give normal consistency. This shows that the physical properties of the mix are not dependent simply upon the ratio of $MgCl_2$ to MgO, but depend also on the amount of water used.

14. Strength and hardness were found to increase with the ratio of cement to aggregate. This increase became less marked as the richness of the mix was increased.

15. The tendency to expand was found to increase with the ratio of cement to aggregate.

16. Tests made on mixes in which the aggregate had been moistened before mixing, in order to secure normal consistency with less magnesium chloride solution, showed that the strength varied directly as the amount of solution used.

17. The time of set was found to be affected by the temperature and humidity, and by the density of the magnesium chloride solution used. High temperature, low humidity, and the use of a dense solution, were found to cause quick setting of the cement.

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APPENDIX

Method for Analysis of Cement

Loss on Ignition—Weigh one gram samples into platinum crucibles, and ignite to constant weight, using Meker burner. The presence of MgO and CaO in the ignited material makes it necessary to use a dessicator containing KOH, in addition to a good drying agent, in order to prevent the material from again taking up moisture and carbon dioxide.

Sulphates—In order to get the sulphates into solution the material is treated with HCl in the usual way, and the sulphates then determined in the most approved manner, by precipitation as $BaSO_4$. It is important to run a blank on the chemicals, and to avoid contamination with sulphur from the flames of gas burners.

The Main Analysis

Solution of Cement—Fuse .5 grams of cement, using 2 grams of pure Na_2CO_3. Dissolve with 10 c. c. of con. HCl, and dilute with water. Evaporate to dryness on the water bath. Dehydrate at 110° for 2 hrs., then add 10 c. c. HCl 1 to 1, digest on the bath for exactly 5 minutes, and filter immediately. Burn the filter paper, ignite the silica, and weigh. Treat the silica with a few drops of H_2SO_4 and determine silica by loss, volatilizing with HF. The non-volatile residue is added to the iron and alumina.

Iron and Aluminium—Dilute the filtrate from the above silica determination to 250 to 300 c. c., add 10 grams NH_4Cl, and precipitate iron and alumina with CO_2 free NH_4OH. Filter and redissolve the Fe and Al, and reprecipitate in the same manner. Ignite and weigh the Fe_2O_3 and Al_2O_3.

Calcium—Acidify the combined filtrates and dilute to 500 c. c. Divide into two equal portions, and reserve one for the subsequent determination of calcium and magnesium. Dilute to 500 c. c., add 1 gram oxalic acid, heat to boiling and make ammoniacal. Boil 10 minutes, and digest on the steam bath for 4-5

hours. Filter and dissolve the calcium with HCl, add .5 grams oxalic acid, and dilute to 300 cc. The solution should now contain about 5 grams NH_4Cl. Boil 10 minutes and digest four hours on the bath, then filter. Wash with hot water, and determine the calcium by titrating with n/10 $KMnO_4$.

Magnesium—The combined filtrates from the calcium are evaporated to a volume of 250 c. c. Acidify with HCl, add 25 cc. of a 10 per cent sol. of microcosmic salt. Heat to boiling, add NH_4OH slowly, and stir; continue until the Mg is practically all precipitated. Add 50 cc. NH_4OH, allow to stand 2–3 hours, and filter. Wash with 2½ per cent NH_4OH. Ignite the precipitate to constant weight, and weigh the $Mg_2P_2O_7$.

Discussion of Methods

Fusion with Na_2CO_3 is recommended, in preference to digestion with HCl, to effect solution of the material, because it brings everything into solution in one operation and eliminates any question in this respect. The objection is sometimes made to the Na_2CO_3 fusion that it introduces a slight error in the calcium and magnesium determinations, due to the occlusion of potassium in their precipitates. However, with the proper dilution, and reprecipitation, there are no grounds for the above objections.

Where the content of magnesia is high, as in these cements, it is very important to dehydrate at a temperature of 110° and never any higher. At higher temperatures magnesium will combine with the silica and thus introduce an error in the silica and magnesium determinations. It is important to determine the silica by volatilization, especially if too high temperatures have been used in dehydration, as this results in the contamination of the silica with considerable amounts of foreign material.

In the precipitation of Fe and Al, it is very important to use a specially prepared ammonia, free from carbonates, or the Fe and Al will be contaminated with $CaCO_3$. Sufficient amounts of NH_4Cl must be present to insure the complete precipitation of aluminum, and to hold the magnesium in solution.

With materials containing extremely large amounts of magnesium, the complete separation of Ca and Mg is impossible, un-

less the proper conditions are obtained. The necessary procedure under such conditions is as previously outlined, it being especially important to secure the large dilution, and the presence of sufficient NH_4Cl. If the dilution is not large enough, or the quantity of NH_4Cl is inadequate, magnesium may contaminate the calcium to a large extent. As a further safeguard, it is well to reprecipitate the calcium.

Because of the high percentage of MgO in magnesia cements, the filtrates from the Fe and Al precipitations are brought up to a known volume, and only one-half taken for the subsequent analysis of calcium and magnesium.

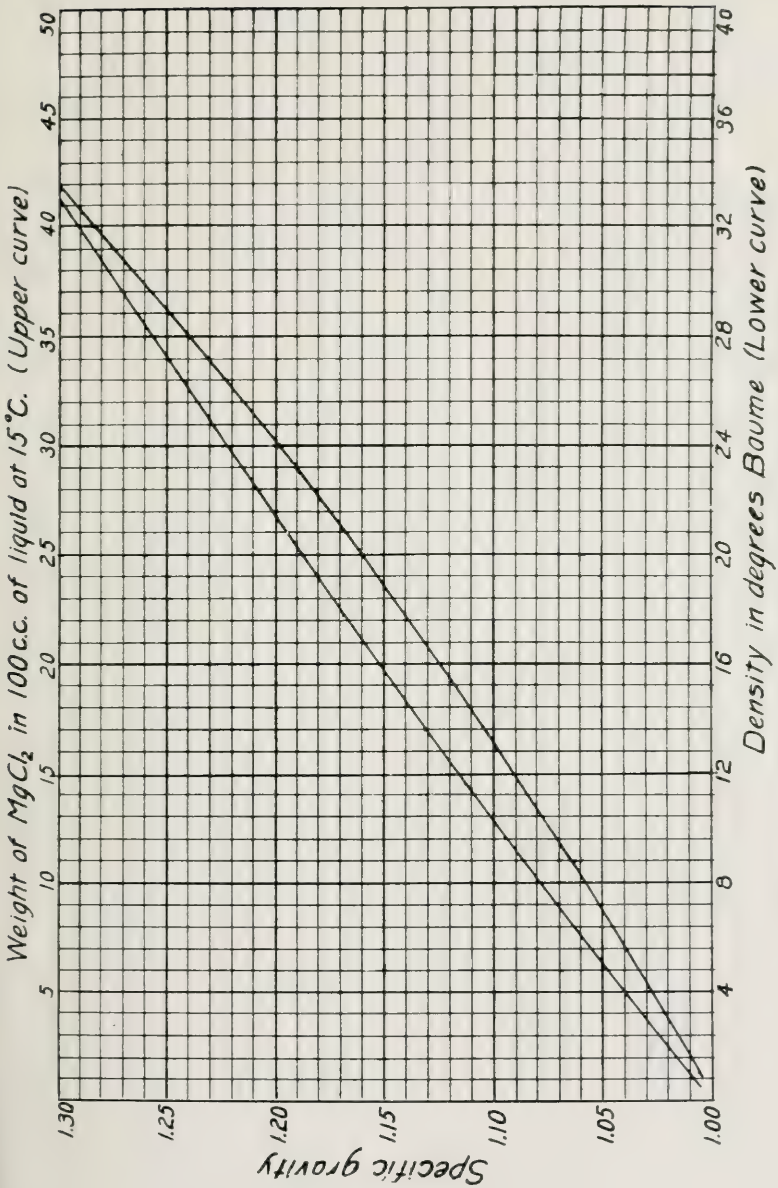

FIGURE 28.—Properties of Magnesium Chloride Solution

BULLETIN OF THE UNIVERSITY OF WISCONSIN

NO. 880

ENGINEERING SERIES VOL. 8, NO. 6, PP. 333-426

A DIGEST OF THE RELATIONS BETWEEN THE ELECTRICAL UNITS, AND OF THE LAWS UNDERLYING THE UNITS

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November, 1917

PREFACE

The features of this digest are stated in the first four sections of the bulletin. In explanation of the purpose in preparing the digest, the following comments are offered:

By far the greater number of practicing electrical engineers and of students graduating in electrical engineering are utterly dependent upon a reference book when it becomes necessary to make any calculations involving units other than the volt, ampere, ohm or watt. This condition is primarily due to the fact that electrostatic theory is offered to the student under one system of units, magnetic theory under a second system, and experience in the laboratory under a third system. To reduce units of the first two systems to the units of the latter system—the so-called practical system—the proper multiple must be selected from the following list:

$$10^{-7},\ 9\times10^{11},\ 10,\ \frac{1}{3\times10^{9}},\ 300,\ 10^{8},\ 10^{9},\ \frac{1}{9\times10^{11}}$$

The result is that the engineer who attaches any quantitative meaning to the theorems of electrostatics and the physicist to whom the data of electrostatics have any work-a-day significance, are rare.

In the writer's estimation the remedy for this deplorable state of affairs is to present electrical theory and data to students in terms of a single system of units which shall include the units of every-day application—the volt, the ampere and the ohm. This digest has been prepared with the object of presenting in a consecutive manner the relations between the units of an ampere, ohm, ampere-turn, weber system of units from which all conversion factors and irrational 4π factors have been eliminated. It is intended that this presentation shall serve as a guide not only for departments of electrical engineering, but also for de-

partments of physics, in presenting the electrical units and definitions to students in electrical engineering.

It is not intended that the information contained in the bulletin about the electrostatic, the electromagnetic, the Heaviside, and the hybrid practical systems, shall be presented to beginning students. This information is primarily for the use of engineers who now carry on their calculations in a mixture of these systems. A student should be burdened with this information only after he has learned to think in terms of a simple factor-free system, and then only because he will find it necessary to read articles written under all of the other systems.

The economy of mental effort which will result from the use of a complete system of electrical units free from conversion factors and irrational 4π factors has long been evident. It is hoped that this digest will be of service—

First. By making it evident to writers who must write in rationalized units that it is unnecessary to use Heaviside electrostatic or electromagnetic units, but that the ampere, ohm, ampere-turn, weber units contain all the elements of a rationalized system.

Second. By showing clearly that the troublesome conversion factors result from the use of units from separate systems in the same calculation (a proceeding which is almost as bad as it would be to express lengths in inches and areas in square centimeters in ordinary calculations), and that these conversion factors can be eliminated from all relations by adhering to the units of the practical system. The conversion factors then become submerged in the constants of materials, which are always taken from tables.

Madison, Wisconsin
January, 1917

CONTENTS

A DIGEST OF THE RELATIONS BETWEEN THE ELECTRICAL UNITS, AND OF THE LAWS UNDERLYING THE UNITS

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1. Features of the Digest—The aim of this digest is to present the relations between the units in the Practical System, and also the relations between the corresponding units in the Practical, Electromagnetic, and Electrostatic Systems, in the form of a **connected development in which the definitions, units, and laws are taken up in the order followed in the development of the electrostatic system of units.** The sequence in which the units are introduced in the electrostatic system, and the fundamental concept in that system—the unit charge—both accord well with the present mode of interpreting electrical phenomena in terms of the properties of the electron—the **atom** or **natural unit** of electricity. On the other hand, magnetic poles and the magnetic properties of permanent magnets are themselves attributed to the motion of electric charges.

2. Other features of the digest are as follows:

a. The equations developed to express the fundamental laws are identically the same for the three systems of units herein discussed, namely, the electrostatic system, the electro-magnetic system, and the practical system of units.

b. All numerical factors, such as 10^{-1}, 10^{8}, 10^{9}, 3×10^{10} and 9×10^{11}, have been banished from these equations by the simple expedient of adhering to the units of a single system in writing an equation. These troublesome factors appear in many of the equations now in use because the prevailing practice is to use mixed units in the same calculation. In this digest, the correct units for calculations in the practical system are emphasized, and their use is strongly advocated.

c. Since the equations involving the units are identically the same in all three systems of units, the name coined for

any unit in the practical system has been applied to the corresponding unit in the electrostatic and electromagnetic systems. All possibility of confusion has been avoided by prefixing E. S. and E. M. to the name of the unit in order to designate units in the latter two systems. Thus, the unit of resistance in the three systems is designated as the ohm, the E. S. ohm, and the E. M. ohm.

d. The equations have been **rationalized.** That is to say, the factor 4π, which in present practice appears in an irrational manner in a number of the most frequently used equations, has been eliminated from these equations. These irrational 4π factors have been banished from the equations by submerging the factors in the empirical constants (of materials) which the equations contain. The rationalization has been accomplished by two expedients:

First: By using the ampere-turn and the ampere-turn per centimeter as the units of magnetomotive force and magnetic intensity.[1] In this digest the ratio of magnetic flux density to magnetic intensity (the former expressed in webers per sq. cm., and the latter in amp-turns per cm.) is termed the **magnetivity** (m) of the medium. Permeability (μ) has been discarded, and the magnetivity (m) is used in all equations. (See Sections 57, 58, and 73.)

Second: By drawing a distinction between the dielectric constant (k) and the permittivity (p) of a dielectric. The permittivity is defined by the equation $k=4\pi p$, and the permittivity (p) is used in all equations.[2] (See Sections 14 and 33.)

By these simple expedients, the substance of all that Heaviside sought to accomplish by his so-called Rational System of Units is realized. That is to say, the irrational 4π factors have been banished from the important equations, and these equations now appear in a form which corresponds with the underlying theory. This method of rationalization involves no legislative action, either by international electrical congresses or by state governments.[3]

e. With the object of clarifying the thought of engineering

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[1] This proposal was first made by Prof. John Perry. See the *Electrician*, p. 355, vol. 27 for 1891.

[2] This expedient was suggested by Prof. R. A. Fessenden. See the *Electrical World*, p. 901, vol. 34 for 1899; also the *Physical Review*, p. 104, vol. 10 for 1900.

students about the relations herein summarized, an attempt has been made to distinguish between the four types of relations summarized; namely, **definitions, experimentally determined relations, deductions,** and **generalizations.** The type to which each relation belongs has been indicated by appending one of the above designations to the name of each relation.

3. Systems of Electrical Units and Their Origin—Electrical phenomena may be divided into two classes: electrostatic phenomena, and electromagnetic phenomena. Under electrostatic phenomena are included all electrical phenomena which may arise from electric charges at rest. Under electromagnetic phenomena are comprised the phenomena related to the motion of electric charges. The motion of the charges may take various forms. It may be: (a) the motion of a charged body of ponderable size, (b) the motion of free electrons and ions through a gas, or an electrolyte, or in space, (c) the motion of electrons through a conductor—the electric current, or (d) the motion of electrons within molecular structures, as in the phenomena of radiation and of magnetism.

Electrostatic and magnetic phenomena were separately discovered and at first were studied independently. Not until Oersted's experiment (1819) showing the deflection of a magnetic needle by a current were they demonstrated to be related. The electrostatic and magnetic fields were

[3] A rationalized system of practical units free of the troublesome decimal factors 10^8, 10^9, 9×10^{11}, etc., may be used in any article, and all uncertainty as to the units may be avoided by a footnote to the following effect:

Footnote—designed to avoid uncertainty as to the units.

All quantities in this article are expressed in that system of practical units in which—

Length is expressed in centimeters.
Mechanical force is expressed in dyne-sevens ($=10^7$ dynes).
Magnetomotive force is expressed in ampere-turns.
Magnetic intensity, H, is expressed in ampere-turns per cm.
Magnetic flux density, B, is expressed in webers per sq. cm. ($=10^8$ maxwells per sq. cm.)
Magnetivity, $m=B/H$, is expressed in webers per sq. cm. per amp-turn per cm.
Electric intensity, F, is expressed in volts per cm.
Displacement, D, is expressed in coulombs per sq. cm.
Permittivity, $p=D/F$, is expressed in coulombs per sq. cm. per volt per cm.
m_0 for free space $=1.257\times10^{-8}$.
p_0 for free space $=8.842\times10^{-14}$.

studied and described by means of the forces exerted respectively on charged bodies and on magnetized bodies placed in the fields. This independent study of the fields from different starting points has given rise to two systems of units for the measurement of electric and magnetic quantities, known as the **electrostatic system** (E. S. S.) or the **electrostatic units** (E. S. U.), and the **electromagnetic system** (E. M. S.) or the **electromagnetic units** (E. M. U.). Since electricity in motion gives rise to a magnetic field, electric quantities and magnetic quantities are not unrelated, but each of these systems may be made complete in itself.

Both the electrostatic and electromagnetic systems give rise to units which were deemed to be either too large or too small for the measurement of the quantities encountered in every-day practice. Accordingly, a committee of the British Association recommended the adoption of a unit of resistance and a unit of voltage which were defined to be 10^9 and 10^8 times as large as the corresponding electromagnetic units.[4] These units were called **practical units.** From these units a third system of units, known as the **practical system** has evolved. The concrete electrical standards legalized throughout the civilized world by governmental action are intended to represent the practical units.[5]

In both the electromagnetic and electrostatic systems of units the centimeter, gram, and second are used as fundamental units. While these three fundamental units suffice for the derivation of all the units required in the measurement of quantities in mechanics, a fourth fundamental unit must be used in the derivation of a system of units for the measurement of electric and magnetic quantities. In the

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[4] Most unfortunately these definitions lead to such a combination of ratios between the practical and the electromagnetic units as 10^{-1}, 10^7, 10^8, and 10^9. As a result, the practical units are not simply related to either the electrostatic or the electromagnetic units.

[5] For a discussion of the practical system of units and a history of the legislative enactments, the following articles may be consulted:

Wolff, Frank A., *The So-called International Units*, Bulletin of the Bureau of Standards, No. 1, Vol. 1 (1904); *The Principles Involved in the Selection and Definition of the Fundamental Electrical Units to be Proposed for International Adoption*, Bulletin of the Bureau of Standards, No. 2, Vol. 5 (1908); *Announcement of a Change in the Value of the International Volt*, Circular 29 of the Bureau of Standards, December, 1910; Dellinger, J. H., *International System of Electric and Magnetic Units*, Scientific Paper No. 292 of the Bureau of Standards, October 11, 1916.

electrostatic system of units, the fundamental definition that of the unit quantity of electricity—implies that the dielectric constant of free space is to be taken as unity. In the electromagnetic system of units, the fundamental definition—that of the unit magnetic pole—implies that the magnetic permeability of free space is to be taken as unity. That is to say: in the electrostatic system, the fourth fundamental unit is the unit **dielectric constant** as represented by the dielectric constant of free space; while in the electromagnetic system, the fourth fundamental unit is the unit of **magnetic permeability** as represented by the permeability of free space.

In the resolutions of the International Electrical Congresses, and in the legislative acts defining and describing the concrete legal standards representing the practical units, the fundamental definitions are those of the concrete standard of resistance, termed the International Ohm, and of the concrete standard of current, termed the International Ampere.[6] The international ohm and the international ampere are intended to equal respectively 10^9 units of resistance and 1/10 of the unit of current in the electromagnetic system. Consequently if in the practical system the equations expressing Ohm's law and Joule's law are to be free of numerical coefficients, the practical unit of energy (termed the joule) and the practical unit of power (termed the watt) must be 10^7 times as great as the corresponding units (namely the erg and the erg per second) in the E. M. and E. S. systems. As a further consequence, if the definitions of force and work—namely,

$$\begin{aligned} \text{force} &= \text{mass} \times \text{acceleration, and} \\ \text{work} &= \text{force} \times \text{distance,—} \end{aligned}$$

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[6] One of these fundamental units alone, in conjunction with the centimeter, gram, and second, is sufficient to establish all the units in the practical system. For example: the **international ohm** having been defined as "the resistance offered to an unvarying electric current by a column of mercury at the temperature of melting ice, 14.4521 grams in mass, of a constant cross-sectional area and of a length of 106.300 centimeters," the **international ampere** might be defined as the unvarying electric current which, when passed through a conductor having a resistance of one international ohm, will result in the expenditure of energy in the conductor at the rate of one joule (defined as 10^7 ergs) per second. The legal definition of the international ampere is, "The International Ampere is the unvarying electric current which, when passed through a solution of nitrate of silver in water, deposits silver at the rate of .00111800 of a gram per second."

are to be preserved free of numerical coefficients, the centimeter and gram cannot both be retained as the remaining two fundamental units of the practical system. If, **as advocated in this digest,** the centimeter and the second are retained as the fundamental units,[7] then the unit of force in the practical system must be the **dyne-seven** ($= 10^7$ dynes), and the unit of mass must be the **gram-seven** ($= 10^7$ grams).

All equations formulated in this digest to express the fundamental laws apply to all three systems of units **provided** the appropriate mechanical units are used. The mechanical units for the E. S. and E. M. systems and for

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[7] It is possible to regard units other than the centimeter, gram-seven, and second as the fundamental units of the practical system. In fact, several writers designate the practical system as the Quadrant, Eleventh-gram, Second (Q.E.S.) System of units (See *Standard Handbook for Electrical Engineers*, 4th Edition, 1915, Section 1, Par. $3d_2$). This designation has its origin in a notion which is thus expressed in Everett's *C.G.S.System of Units*, 5th edition, page 217: "The Practical Units are the units which would be obtained by direct derivation from the fundamental units T =1 sec. L =10^9 cm. (=1 earthquadrant), M =10^{-11} grams (=1 eleventh-gram.)" This statement fails to express the entire thought of its author. To make the statement complete, the following clause should be added to it: **provided, further, that the value unity be assigned to the magnetic permeability of air (μ_o) occurring in the expression for the force between two magnetic poles—namely, f =** $\frac{s_1 s_2}{\mu_o l^2}$

In the writer's estimation, the designation **Quadrant, Eleventh-gram, Second** for the practical system of units is productive of confusion and is objectionable. It is productive of confusion because the name leads students to attempt to formulate a practical system of units based on the quadrant as the unit length. A few of the units in such a system have been tabulated in the last column of Table I. It is improbable that anyone ever attempted to think in such units. The designation is objectionable because it implies that the quadrant must necessarily be taken as the unit of length in the practical system. There is no necessity for basing the practical electrical units on the quadrant. This is evidenced by the Meter, Kilogram, Second (M. K. S.) System formulated by Prof. Giorgi, and by the Centimeter, Gram-seven, Second (C. G. S. S.) System herein formulated.

Prof. G. Giorgi has advocated the use of the meter, the kilogram and the second as the mechanical units of the practical system. (See M. Ascoli, *On the Systems of Electric and Magnetic Units*, Transactions of the International Electrical Congress at St. Louis, 1904; page 136, vol. I.) A few of the units of the Giorgi (M. K. S.) system have been tabulated in the third column of Table I. In the estimation of the writer, the M. K. S. system labors under a disadvantage in that its use would entail a departure from the well established practice of thinking and writing of **flux densities per square centimeter** and of **gradients per centimeter.** In the case of such an important and ever present quantity as length, a great deal is gained by adhering in the Practical System to the unit used in the E. S. and E. M. systems—namely the centimeter. To illustrate this point, a few of the practical units (C. G. S. S.) based upon the centimeter as the unit of length have been tabulated with the corresponding M. K. S. and Q. E. S. units in Table I.

the practical system used in this digest (the C. G. S. S. system) have been tabulated in Table II.[8]

TABLE II

MECHANICAL UNITS IN THE SYSTEMS OF ELECTRICAL UNITS

Quantity	E. M. and E. S. Units	Practical Units
Length	cm.	cm.
Time	second	second
Mass	gram	gram-seven
Velocity	cm. per sec.	cm. per sec.
Force	dyne	dyne-seven
Energy	erg	joule=10^7 ergs
Power	erg per sec.	watt.=10^7 ergs per sec.

TABLE I

A COMPARISON OF PRACTICAL UNITS BASED UPON THE CENTIMETER, THE METER, AND THE QUADRANT

Quantity	C. G. S. S. Units herein advocated	M. K. S. Units Giorgi Units	Q. E. S. Units
Length	cm.	meter	10^9 cm.
Time	second	second	second
Resistance	ohm	ohm	ohm
Current	ampere	ampere	ampere
Energy	joule	joule	joule
Force	dyne-seven	dyne-five	centi-dyne
Mass	gram-seven	gram-five	eleventh-gram
Current density	amp. per sq. cm.	amp. per sq. m.	amp. per. sq. qd.
Flux density	webers per sq. cm.	webers per sq. m.	webers per sq. qd.
Resistivity	ohm-cm.	ohm-meter	ohm-quadrant
Electric intensity	volts per cm.	volts per meter	volts per qd.
Magnetic intensity	amp-turn per cm.	amp-turn per m.	amp-turn per qd.
Constants of space			
Dielectric constant k_o	1.111×10^{-12}	1.111×10^{-10}	1.111×10^{-3}
Permeability μ_o	10^{-9}	10^{-7}	1.00
Permittivity p_o	8.842×10^{-14}	8.842×10^{-12}	8.842×10^{-5}
Magnetivity m_o	1.257×10^{-8}	1.257×10^{-6}	12.57
Velocity of light	3.0×10^{10}	3.0×10^{8}	$30.$

[8] In none of the engineering literature which has fallen under the writer's observation, except Karapetoff's *Magnetic Circuit*, has the suggestion been made that the unit of mechanical force in the practical system is the dyne-seven. The general

Names have been coined for the units in the practical system only.[9] The same names may be applied to great advantage to the corresponding electrostatic and electromagnetic units, provided the units in the latter two systems be designated by attaching suitable prefixes to the names of the practical units. In this digest, the E. S. and E. M. units are designated by prefixing E. S. and E. M. to the names of the practical units.[10] Thus, the unit of resistance is designated in the three systems as the ohm, the E. S. ohm and the E. M. ohm.

4. Classification of Relations.—In the following digest there are relations of four types:

1. Relations which are matters of **definition.**

2. Relations which have been **determined** purely by observation and **experiment.**

3. Relations which have been **deduced** from fundamental experimental relations for the values of newly defined quantities.

4. Relations which are **generalizations** embodying and based upon general experience or upon extended experimental evidence. Generalizations are not always susceptible of direct proof: witness, the principle of the conservation of energy, and the second law of thermodynamics.

Notwithstanding the brevity of the time which has elapsed since the master minds formulated the principles underlying electrical measurements, the notions of the latest disciples not infrequently exhibit a mythological growth as gross and misleading as that surrounding any ancient rite. The essence of this idolatry[11] is the failure to recognize the part

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practice in calculating the mechanical forces arising from electric charges and currents seems to be, either (a) to reduce the quantities to E. S. or E. M. units and to calculate the force in dynes by means of the appropriate E. S. or E. M. formula, or (b) to compute the energy transformation which would result from a small movement between the bodies and from this to deduce an expression for the mechanical force in dynes, kilograms, or pounds There is a distinct step from these methods to the recognition that mechanical forces can be directly and expeditiously calculated **in practical units provided the dyne-seven is recognized as the practical unit of force.**

[9] There are four exceptions to this statement. **Gauss, oersted, gilbert,** and **maxwell** have been assigned as names to four of the electromagnetic units.

[10] Another practice is to designate the E. S. and E. M. unit by prefixing **stat** and **ab** respectively to the names of the practical units. Thus, the ohm is designated in the three systems as the ohm, the statohm, and the abohm.

[11] See Max Müller, *Science of Language*, vol. II, Chaps. 9 and 13; also Bacon, *Novum Organum*, Book I, Aphorisms 43, 59, and 60.

played by definition in so-called demonstrations, and the consequent failure to recognize that certain relations are purely matters of definition or convention.

The recognition of these four types is of such importance that the type under which each of the following relations falls has been indicated by appending to the name of the relation either [**Definition**], [**Exp. Det. Rel.**] (for experimentally determined relation), [**Deduction**] or [**Generalization**].

It is presumed that the reader is familiar with the experiments upon which the ideas involved in the following definitions are based, and from which the laws have been derived.

5. Like and Unlike Electricities: [**Definition**]—Two electrified bodies are said to be charged with **like electricity** if they act in like manner toward a third charged body; that is, if both repel or both attract a small charged test body. They are said to be charged with **unlike electricities** if one repels and the other attracts the charged test body.

Positive and Negative Electricities: [**Definition**]—Unlike charges tend to neutralize each other's effects, therefore the two kinds of electricity are named positive and negative electricities, or positive and negative charges. A charge like that acquired by a glass rod which has been rubbed with a silk cloth is arbitrarily called a **positive** charge, and a charge like that acquired by the silk is called a **negative** charge.

6. Two Kinds of Electricity Only: [**Exp. Det. Rel.**]—Under a classification based upon the nature of the forces (whether attractive or repulsive) exhibited between charged bodies, there are but two kinds of electricity.

7. First Law of Electrostatics: [**Exp. Det. Rel.**]—Bodies charged with like electricity repel each other; bodies charged with unlike electricity attract each other.

8. Equal Quantities of Electricity: [**Definition**]—Two bodies are said to be charged with equal quantities of electricity of like sign if they produce the same effect on the

electrification of an insulated hollow conducting vessel, as a hollow metallic sphere, when they are separately and successively placed within the vessel. Two bodies are said to be charged with equal quantities of electricity of opposite sign if they produce no effect on the electrification of the vessel when simultaneously introduced.

9. Positive and Negative Charges are Always Developed Simultaneously in Equal Amounts: [Generalization]—When a separation of charges is brought about by any means, equal quantities of positive and negative electricity are always developed. When electrical charges combine, equal quantities of positive and negative electricity disappear. That is, the two electricities always exist in equal quantities.

10. The Electric Field: [Definition]—The region surrounding a charged body, or in general, any region in which a charged body experiences a mechanical force by virtue of its charge, is called a **field of electric force** or an **electric field.** In setting up an electric field, work must be done in bringing about a separation of the charges against their mutual attraction. The energy so expended is not dissipated, but is stored. It may be converted back to the mechanical form by allowing the charges to do work in pulling together again against resisting forces. The work done in separating the charges is said to be converted into "electro-potential" energy, and is conceived to be stored in the surrounding medium by reason of "constraints" set up in the medium. It will develop that the state of the medium at any point may be specified by means of two quantities termed the **electric intensity** and the **displacement** or **electrostatic flux density.**

11. Coulomb's Law of Force between Charged Bodies: [Exp. Det. Rel. (1787)]—The force of repulsion (f) between two charged bodies in an infinitely extended homogeneous medium, far apart as compared with their dimensions, is proportional to the product of their charges and inversely proportional to the square of the distance (l)

between their centers, and it depends upon the dielectric, or insulating medium, in which the bodies are immersed.[12]

$$f\ (\text{repulsion}) = \frac{Q_1Q_2}{kl^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

12. Dielectric Constant and Specific Inductive Capacity: [**Definitions**]—The constant (k) appearing in the denominator of the expression for the force between charged bodies, is called the **dielectric constant** of the medium or substance in which the bodies are immersed. (k) is conceived to be a constant of the medium;[13] its value will depend upon the units in which force, length, and **quantity of electricity** are expressed. On the other hand, if the units of length and force alone are specified, any numerical value may be arbitrarily assigned to the dielectric constant of some specified medium. This is equivalent to defining unit dielectric constant to be 1/k th as great as the dielectric constant of the specified medium. In the electrostatic system of units, the fundamental definition—that of the unit quantity of electricity—implies that the dielectric constant of free space is to be taken as unity.

The dielectric constant of free space when expressed in E. M. units has the numerical value $\frac{1}{9\times10^{20}}$; and when expressed in practical units, it has the numerical value $\frac{1}{9\times10^{11}}$. The value $\frac{1}{9\times10^{20}}$ is to be regarded as an experimentally determined constant of space, whose value is fixed by the manner in which unit magnetic pole and unit current have been defined in the E. M. system. Likewise the value $\frac{1}{9\times10^{11}}$ is to be regarded as an experimentally determined

[12] This relation was experimentally determined in 1787 by Coulomb. However, the fact that the force between two charges depends upon the medium was unknown (except to Cavendish, who noted this fact about 1776 but failed to disclose it) until Faraday discovered the effect of the medium in 1837.

[13] This conception of (k) must be abandoned in that mode of interpretation in which the difference in the properties of mediums—such as glass, oil, and air—is accounted for by the elastic displacement of electric charges in the medium. In this case, equation (1) expresses a relation pertaining to the ether which pervades all substances, and (k) is to be regarded as a constant whose value depends only upon the units in which force, length, and quantity of electricity are expressed.

constant of space whose value is fixed by the manner in which the unit of resistance and the unit of current have been defined in the practical system.

$$\text{The practical unit dielectric constant equals} \begin{cases} 9 \times 10^{11} \text{ E. S. units} \\ 10^{-9} \text{ E. M. units} \end{cases}$$

The **relative dielectric constant** k_r of a substance—also called its **specific inductive capacity** is defined as the ratio of its dielectric constant to the dielectric constant of the standard medium, free space. (For air, k has the value 1.00059.)

13. Unit of Electricity: [Definition]—The assignment of a numerical value to the dielectric constant of a specific medium at once fixes the unit of electricity. The unit must be so defined as to satisfy equation (1). Thus—

The **unit of electricity** is that quantity of electricity with which a very small body must be charged so that when placed in a vacuum at unit distance (one centimeter) from a similar body charged with an equal quantity, the force of repulsion between them will be 1/k times the unit force.

$$1/k \text{ times the unit force is} \begin{cases} \text{one dyne in the E. S. system} \\ 9 \times 10^{20} \text{ dynes in the E. M. system} \\ 9 \times 10^{11} \text{ dyne-sevens } (9 \;\; 10^{18} \\ \quad \text{dynes) in the practical system} \end{cases}$$

In the practical system, the unit quantity of electricity is called the **Coulomb.**

$$\text{The practical coulomb equals} \begin{cases} 3 \times 10^{9} \text{ E. S. coulombs} \\ 10^{-1} \text{ E. M. coulombs} \end{cases}$$

14. Permittivity of a Medium: [Definition]—(See Section 33)—To those responsible for the formulation of our systems of units, it seemed wise to express Coulomb's inverse square law by the simple equation (1), and to assign the value unity to the dielectric constant (k_0) of free space. In the further development of electrical theory, this constant k is found to appear in many other formulae. Most unfortunately, the factor 4π, or its reciprocal, appears in many of these formulae in a manner which has been characterized as puzzling and irrational. (See Section 33.) It develops

that these irrational 4π factors creep into the important and most frequently used equations because of the manner in which the unit quantity of electricity and the subsequent units have been fixed by equation (1). It is evident that a simple expedient will suffice to banish the 4π from those equations involving the factor $\frac{k}{4\pi}$. This expedient is to discard the dielectric constant k and to rewrite the equation expressing the inverse square law in the form

$$f = \frac{Q_1 Q_2}{4\pi p l^2} \qquad (2)$$

In other words, the irrational and objectionable 4π may be banished from the formulae most frequently used in engineering calculations by shifting it to a formula which is little used. The attempt to explain the appearance of the 4π in equation (2) can never be perplexing in the same sense that it is perplexing in the case of the formulae to be subsequently developed. The 4π appears in equation (2) because we deliberately put it there, and we put it there to escape what Heaviside has termed an eruption of 4π's in subsequent equations.

The constant p appearing in the denominator of expression (2) for the force between charged bodies is termed the **permittivity** of the medium or substance in which the bodies are immersed.

In the subsequent pages we propose to express Coulomb's inverse square law by equation (2), and we propose to use the **permittivity** p rather than the **dielectric constant** k in all formulae.

If the unit quantity of electricity defined by equation (2) is to be the same as that defined by equation (1), the numerical values of the dielectric constant k and of the permittivity p of any medium must be related to each other as expressed in equation (3).[14]

$$k = 4\pi p \qquad (3)$$

[14] The distinction which is here drawn between the dielectic constant and the permittivity is not universally observed. The two terms are frequently used synonymously.

Heaviside was the first to point out that the 4π's might be more advantageously distributed among the equations (he termed this **rationalizing** the equations)

From this it follows that the permittivity p_o of free space has the following values in the three systems:

$$\text{The permittivity}^{15}\text{ of free space, } p_o\text{, equals} \begin{cases} .0796 & \text{in the E. S. system} \\ 8.84 \times 10^{-23} & \text{in the E. M. system} \\ 8.84 \times 10^{-14} & \text{in the Practical system} \end{cases}$$

$$\text{The practical unit of permittivity equals} \begin{cases} 9 \times 10^{11} & \text{E. S. units} \\ 10^{-9} & \text{E. M. units} \end{cases}$$

No short name has been assigned to the unit of permittivity. See Section 33 for a name for the unit,—a name based upon the appearance of p as a proportionality constant between other quantities.

The relative permittivity (p_r) of a substance is defined as the ratio of the permittivity of the substance to the per-

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by expressing Coulomb's inverse square law by an equation of the form (2). (See *Electrician*, Vol. 10, 1882, p. 6; *Electrical Papers*, Vol. 1, pp. 199, 262, 432; Vol. 2, pp. 543, 575; *Electromagnetic Theory*, Vol. 1, p. 116.) In his system of so-called Rational Electrostatic Units (R. E. S. U.), Heaviside assigned the value unity to the permittivity of free space. This is equivalent to defining unit charge as a charge of such magnitude that in free space it repels an equal charge at unit distance with a force of $1/4\pi$ dynes. Consequently the R. E. S. coulomb equals $\frac{1}{\sqrt{4\pi}}$ E.S. coulombs.

The advantage (or disadvantage) of the Heaviside Rational Electrostatic System is that the assignment of the value unity to the permittivity of free space has a tendency to suppress the constant p in calculations having to do with conditions in space or in air.

As pointed out by Fessenden, the assignment of the value unity to the permittivity of space is not an essential part of the program of rationalizing the equations. The legalized units of current resistance, voltage, etc., may be left undisturbed and the important equations may be rationalized in the following manner:

a. Adopting the first half of Heaviside's proposal—namely, expressing Coulomb's law by a formula in the form of equation (2).

b. Assigning to the constant p appearing in equation (2) a name which will serve to distinguish it from the constant k appearing in equation (1). Fessenden proposed to call p the **capity**, but Heaviside's term **permittivity** has prevailed.

c. Determining the value to be assigned to the new constant p by equation (3).

d. Expressing magnetomotive force in ampere-turns, as proposed by Perry, instead of in gilberts.

(See *Electrical World*, Vol. 34, 1899, p. 901; and Vol. 35, 1900, p. 282.)

[15] The permittivity p_o of free space when expressed in the Heaviside Rational Electrostatic Units (H. R. E. S. U.) is, by definition, unity: when expressed in the Heaviside Rational Electromagnetic Units (H. R. E. M. U.), it is $\frac{1}{9 \times 10^{20}}$

The Heaviside R. E. S. coulomb equals $\frac{1}{\sqrt{4\pi}}$ E. S. coulombs

The Heaviside R. E. M. coulomb equals $\frac{1}{\sqrt{4\pi}}$ E. M. coulombs.

mittivity of the standard medium, free space. (For air, p_r has the value 1.00059.) The relative permittivity of any substance is, of course, equal to its relative dielectric constant or its specific inductive capacity.

15. Superposition of Electric Fields: [Generalization]—The force exerted upon a charge at a point P, which is at a distance from a number of concentrated charges, is found to coincide with the calculated force obtained by determining the force at P which each charge would give rise to if it alone were in the field, and then, by the polygon of forces, calculating the resultant of all these forces.

16. Potential Difference and Potential at a Point: [Definitions]—The **potential difference** (E) between two points B and C is defined as the work which would be done by the field per unit charge upon an indefinitely small positively charged body as the body moves from B to C—the body and its charge both being so small that they produce no sensible alteration in the previous distribution of the charges. The potential at B is higher than at C if the field does work upon a positive charge which moves from B to C.

The potential (E) at any point in the field is defined as the work which would be done by the field per unit charge upon an indefinitely small positively charged body as the body moves from the given point to an infinite distance from the charges giving rise to the field—the body and its charge both being so small that they produce no sensible alteration in the previous distribution of the charges. The potential at a point is the potential difference between the point and points at an infinite distance from the charges.

The **unit of potential difference** in the practical system is called the **volt.** The potential difference between two points is one **volt** if the work done by the forces of the field per **coulomb** of positive electricity which passes from the point of higher to the point of lower potential is one **joule** ($=10^7$ ergs).

$$\text{The practical volt equals} \begin{cases} \dfrac{1}{300} \text{ E. S. volts} \\ 10^8 \text{ E. M. volts} \end{cases}$$

$$E \text{ (increase from B to C)} = \frac{\text{Work done on dQ as it moves from B to C}}{dQ}$$

$$E \text{ (increase from B to C)} = -\frac{dA}{dQ} \quad \ldots\ldots\ldots\ldots (4)$$

$$E \text{ (increase from B to C)} = -\int \mathbf{F} \cos\theta \, dl \quad \ldots (4a)$$

in which

F is a vector representing at any point P the magnitude and direction of the force exerted per unit + charge upon an infinitesimal charge at the point P,

and θ is the angle between the vector **F** and the tangent to the path BC at the point P.

From equation (4a), the **potential gradient** $\frac{dE}{dl}$ in a given direction is equal and opposite in sign to the component in the given direction of the force exerted per unit positive charge.

$$\frac{dE}{dl} = -\mathbf{F} \cos\theta \quad \ldots\ldots\ldots\ldots (4b)$$

From the definition of potential difference, it follows that if a total quantity of Q coulombs moves from a point of higher potential to a point lower in potential by E volts, the work A done by the forces of the field is—

$$A = EQ \quad \ldots\ldots\ldots\ldots (4)$$

17. Potential Due to One or More Concentrated Charges: [**Deductions**]—The potential E at a point P which is at a distance l_1 from a concentrated charge of Q_1 coulombs may be deduced from Coulomb's law thus:

$$E = \frac{\text{Work done in repelling a charge dQ}}{dQ} =$$

$$\frac{1}{dQ}\int_{l_1}^{\infty} \frac{Q_1\,(dQ)}{4\pi p l^2}\,dl = \left[-\frac{Q_1}{4\pi p l}\right]_{l_1}^{\infty}$$

$$E = \frac{Q}{4\pi p l_1} \text{ volts} \quad \ldots\ldots\ldots\ldots (5)$$

From the fact that fields may be superposed in the manner stated in Section 15, it follows that the potential at a point P which is at the distances l_1, l_2, etc., from the charges Q_1, Q_2, etc., is given by the expression

$$E = \frac{1}{4\pi p}\left[\frac{Q_1}{l_1} + \frac{Q_2}{l_2} + \frac{Q_3}{l_3} + \right] \qquad \text{(5a)}$$

18. Electric Intensity, or Electric Force, or Potential Gradient: [**Definition**]—The electric intensity, or electric force, at any point in the field is defined as a vector $\mathbf{F}$ whose magnitude is equal to the force per unit charge with which a small positively charged body would be acted upon,—the body and its charge both being so small as not to sensibly alter the previous distribution of the charges; the direction of the vector is to coincide with the direction of the force on the small positive charge.

$$\mathbf{F} = \frac{d\mathbf{f}}{dQ} \qquad \text{(6)}$$

The electric intensity at any point is a vector which is equal and opposite in direction to the vector representing **the** (maximum) potential gradient at the point (See equation 4b).

$$\mathbf{F} = -\left[\frac{dE}{dl}\right]_{max} \qquad \text{(4c)}$$

Unit electric intensity is such an intensity of field that unit charge is acted upon with unit force. This unit is termed an electric intensity, or potential gradient, of **one volt per cm.**

19. Mechanical Force Acting upon a Charge in a Field: [**Deduction**]—From the definition of electric intensity, it follows that the mechanical force $\mathbf{f}$ exerted by the field on a concentrated charge Q situated at a point where the electric intensity is $\mathbf{F}$ is expressed by the equation

$$\mathbf{f} = Q\mathbf{F} \qquad \text{(6)}$$

20. Electric Intensity due to a Concentrated Charge: [**Deduction**]—The electric intensity $\mathbf{F}$ at a point P which

is at a distance from a concentrated charge of Q coulombs, by Coulomb's law, is

$$\mathbf{F} = \frac{Q}{4\pi p l^2} \text{ volts per cm.} \qquad (7)$$

$\mathbf{F}$ is directed radially away from the charge Q.

21. Lines and Tubes of Electric Intensity or Force: [Definitions]—If a curve is drawn in the field so that at every point in the field its direction coincides with the force which would be exerted upon a small charged body at the point, the curve is called a **line of electric intensity,** or of **electric force.** A **tube of force** is the space enclosed by the surface formed by drawing lines of force through every point of a small closed curve. The positive direction along a line of force is defined to be the direction in which a positive charge tends to move.

22. Relation between the Potential Difference between Two Conductors and the Charge on the Conductors: [Exp. Det. Rel.] Condenser: [Definition]—If a charge is transferred from an insulated conductor or connected set of conductors A to another insulated set of conductors B, a difference of potential is thereby set up between the conductors A and B. It has been experimentally determined that for a given system of conductors the potential difference E between the sets of conductors is directly proportional to the quantity of electricity Q which has been transferred.

$$\frac{Q}{E} = C \text{ (a constant)} \qquad (8)$$

If the two conductors A and B have extended surfaces separated by a thin layer of the insulating medium, the constant C, or the ratio of Q to E, is large. Such an arrangement of conductors and insulating medium is called a **condenser.**

The carrying of a positive charge from plate A to plate B of a condenser leaves an equal negative charge on plate A. By the quantity of electricity in a condenser, or the charge in a condenser, is meant—not the sum of the two charges—

but the quantity of electricity which has been transferred from one plate to the other.

23. Permittance, or Capacity, and Elastance of a Condenser: [**Definition**]—The proportionality constant between the charge Q and the potential difference E, equation (8), is termed the **permittance** or **capacity** of the two systems of insulated conductors with reference to each other, and is symbolized by C.

$$C = \frac{Q}{E} \quad \text{...} (8)$$

The permittance or capacity of a condenser is thus defined as numerically equal to the charge which must be transferred from one conductor to the other to cause unit difference of potential between the conductors. The unit of capacity is termed the **farad;** a condenser has a capacity of one farad if a charge of one coulomb causes a difference of potential of one volt.

By the **elastance** S of a condenser is meant the reciprocal of its permittance. The elastance is therefore the ratio of the potential difference to the charge in the condenser. The unit of elastance is termed the **daraf.** A condenser with a permittance of one farad has an elastance of one daraf.

$$S = \frac{E}{Q} \quad \text{...} (8a)$$

$$\text{The practical farad equals} \begin{cases} 9 \times 10^{11} \text{ E. S. farads} \\ 10^{-9} \text{ E. M. farads} \end{cases}$$

24. Mutual Elastance between Two Condensers: [**Definition**]—Imagine four conductors 1a, 1b, 2a, and 2b to be insulated from each other. Let conductors 1a and 1b be regarded as constituting Condenser 1, and let 2a and 2b be regarded as constituting Condenser 2. Suppose the four conductors are so located with reference to each other that any charge in one condenser causes a difference of potential between the conductors constituting the other condenser.

Let E_1 represent the difference of potential between the conductors 1a and 1b which results from the charge Q_2 in Condenser 2, and E_2 represent the difference of potential

between the conductors 2a and 2b which results from the charge Q_1 in Condenser 1. It may be shown that the ratio of E_1 to Q_2 is equal to the ratio of E_2 to Q_1.

$$\frac{E_1}{Q_2} = \frac{E_2}{Q_1} \qquad \text{(9a)}$$

By the mutual elastance S_m of the two condensers is meant the difference of potential between the plates of one condenser which results from unit charge in the second condenser.

$$S_m = \frac{E_1}{Q_2} = \frac{E_2}{Q_1} \qquad \text{(9)}$$

25. Energy Stored in a Condenser: [Deduction]—The work done in conveying a charge from one conductor of a condenser to the other is stored in the condenser. The energy thus stored in charging a condenser may be computed in the following manner:

Let C represent the permittance of the condenser, and
q " the quantity of electricity in the condenser at any instant.

Then, the difference of potential (e) between the conductors is given by the equation (8) as—

$$e = \frac{q}{C}$$

If, now, the charge (q) is increased by the infinitesimal amount (dq) the work (dW) which is done in transferring the charge (dq)—or the energy which is thereby stored in the condenser—is (equation 4) the product of the potential difference (e) between the conductors times the quantity transferred.

$$dW = e(dq) = \frac{q}{C}(dq)$$

The total energy W stored in a condenser of permittance C having a charge Q in the condenser may be found by integrating the above expression between the limits O and Q.

$$W = \int_0^Q \frac{1}{C} q(dq) = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} SQ^2 \qquad \text{(10a)}$$

If the difference in potential between the conductors with the charge Q in the condenser is represented by E, then by equation (8) $Q = CE$. Therefore the expression for the energy may be also written in the forms,

$$W = \tfrac{1}{2}CE^2 = \frac{1}{2}\frac{E^2}{S} \quad \text{.........................(10b)}$$

and $$W = \tfrac{1}{2}QE. \quad \text{.........................(10c)}$$

26. Energy Stored in Two Condensers having Mutual Elastance: [**Deduction**]—Let the two condensers be numbered (1) and (2), and let the following equations express the relations between the induced differences of potential and the charges:

$$E_2 = S_{m1}Q_1$$
$$E_1 = S_{m2}Q_1$$

By an argument in which the two condensers are carried through a complete cycle of changes, the two constants S_{m1} and S_{m2} may be shown to be equal. The steps in the argument are as follows:

Initial condition: Let both condensers be initially uncharged.

Step 1: Let the quantity of electricity Q_1 be transferred (in infinitesimal amounts) from one plate to the other of Condenser 1. The work done and the energy thereby stored is $\frac{1}{2}\, S_1\, Q_1^2$.

Step 2: Let the quantity Q_2 be now transferred in infinitesimal amounts from one plate to the other of Condenser 2. The work done in transferring the charge, and the energy thereby stored is—

$$W = \int_0^{Q_2} (S_{m1}Q_1 + S_2Q)\, dQ = S_{m1}Q_1Q_2 + \tfrac{1}{2}S_2Q_2^2$$

Step 3: Let Condenser 1 be now discharged by transferring the quantity Q_1 in the opposite direction. The work done by the forces of the field, or the energy thereby returned is—

$$W = \int_0^{Q_1} (S_{m2}Q_2 + S_1Q)\, dQ = S_{m2}Q_1Q_2 + \tfrac{1}{2}S_1Q_1^2$$

Step 4: Let Condenser 2 be now discharged. The energy thereby returned is $\frac{1}{2} S_2Q_2^2$.

Conclusion: The condensers are now in their initial condition. The energy delivered to the condensers during the cycle is—

$$\tfrac{1}{2}S_1Q_1^2 + S_{m1}Q_1Q_2 + \tfrac{1}{2}S_2Q_2^2$$

The energy returned by the field during the cycle is—

$$\tfrac{1}{2}S_1Q_1^2 + S_{m2}Q_1Q_2 + \tfrac{1}{2}S_2Q_2^2$$

If these two quantities are not equal, this cycle, or the reverse of this cycle, will violate the principle of the conservation of energy. Therefore, **S_{m1} must equal S_{m2}**, and the subscripts $_1$ and $_2$ may be discarded.

The energy stored in the two condensers having the charges Q_1 and Q_2 is—

$$W = \tfrac{1}{2}S_1Q_1^2 + S_mQ_1Q_2 + \tfrac{1}{2}S_2Q_2^2 \qquad (11)$$

27. Mechanical Force between Two Charged Conductors Constituting a Condenser: [Deduction]—Let the capacity of the conductors with reference to each other be represented by C, and let the quantity in the condenser be Q. Imagine the conductors to be isolated so that the quantity Q remains constant, and suppose the distance between the conductors to be changed by the infinitesimal amount (dx) by a displacement of one of the conductors. This displacement is to be a pure translation without rotation. (The distance (dx) is measured in the direction of translation, and is always taken as a positive quantity.)

Imagine the displacement of the conductor to cause an **increase** in the capacity by the amount (dC).

The energy stored before the displacement was $\frac{1}{2}\frac{Q^2}{C}$

The energy stored after the displacement is $\frac{1}{2}\frac{Q^2}{(C+dC)}$

The decrease in the stored energy is $\frac{1}{2}\frac{Q^2}{C} - \frac{1}{2}\frac{Q^2}{(C+dC)}$

$$= \frac{1}{2}\frac{Q^2}{C^2}dC$$

By the principle of the conservation of energy, this decrease in the stored electrical energy must equal the mechanical work done by the electrical system when the conductor moves over the distance (dx). If (f) represents the **component-of-the-force** on the displaced conductor (arising from the charges) tending to move the conductor **in the direction of the displacement (dx) from the initial to the final position,** the mechanical work done by the conductor is f(dx).

$$\text{Therefore } f\,(dx) = \frac{1}{2}\,\frac{Q^2}{C^2}\,dC$$

$$\left.\begin{aligned} \text{or } f &= \frac{1}{2}\,\frac{Q^2}{C^2}\,\frac{dC}{dx} \\ &= \tfrac{1}{2}\,E^2\,\frac{dC}{dx} \end{aligned}\right\} \qquad (12)$$

28. The Dielectric Circuit—Permittance of a Parallel Plate Condenser: [Exp. Det. Rel.] Energy Stored in Unit Volume of the Dielectric: [Deduction]—The expressions for the permittance in terms of the dimensions, and for the energy stored per unit volume of the dielectric, may be readily deduced for the case of a condenser consisting of two extended parallel plates separated by a distance (l) which is small in comparison with the length and breadth of the plates. If this condenser is charged, the surface density of the charge—that is, the quantity of electricity per unit area—will be uniform all over the adjacent surfaces except near the edges of the plates.

Let the quantity of electricity per unit area of the positive plate be represented by (+q). Then the quantity per unit area of the negative plate is (−q). To calculate the potential of a point P on the positive plate, imagine the surface of the positive plate to be divided into elementary circular strips of width (dx), by circles drawn with P as a center. Similarly, let the negative plate be divided by circles drawn around the point P^1 corresponding to point P.

The charges on the two circular strips of radius (x) are 2π x dx q and -2π x dx q, and their respective distances

from P are (x) and $\sqrt{x^2+l^2}$. These charges would cause at P a potential (dE) which, from equation (5a), is—

$$dE = \frac{q 2\pi\, x\, dx}{4\pi p x} + \frac{-q\, 2\pi\, x\, dx}{4\pi p \sqrt{x^2+l^2}}$$

The potential at P due to the entire charge on the two plates is found by summing up the effect of all the circular strips,

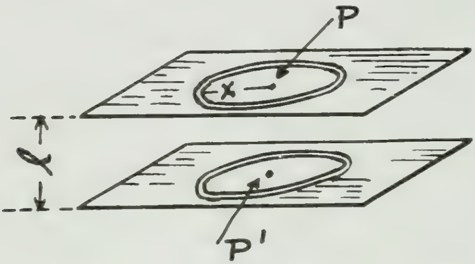

FIG. 3

or by integrating the above expression between the limits $x=0$ and $x=\infty$.

$$E_p = \frac{q}{2p}\int_0^\infty dx - \frac{x dx}{\sqrt{x^2+l^2}} = \frac{q}{2p}\Big[x - \sqrt{x^2+l^2}\Big]_0^\infty$$

$$E_p = \frac{ql}{2p}$$

In like manner, the potential of the negative plate is shown to be—

$$E_n = -\frac{ql}{2p}$$

Therefore the difference in potential between the two plates is—

$$E = E_p - E_n = \frac{ql}{p}$$

Let any cylindrical portion of this condenser, far removed from the edges and comprised between two portions of the parallel plates opposite each other and each of area α, be

now considered. The quantity of electricity on the area α of the positive plate is αq, therefore the expression for the quantity of electricity Q in this portion of the condenser is [16]—

$$Q = p\frac{\alpha}{l}E \qquad (13)$$

and therefore the permittance C of this portion of the condenser is [16]—

$$C\left(\text{or } \frac{Q}{E}\right) = p\frac{\alpha}{l} \qquad (14)$$

The energy W stored in this portion of the condenser is—

$$W\left(=\tfrac{1}{2}QE\right) = \tfrac{1}{2}p\frac{E}{l}\alpha E \qquad (15a)$$

Since E is the potential difference between **parallel** plates and l is the distance between the plates, E/l is the potential gradient or electric intensity $\mathbf{F}$ in the dielectric. Hence, the expression for the stored energy may be written—

$$W = \tfrac{1}{2}(pF)\alpha E \qquad (15b)$$

This energy is conceived to be stored in the dielectric between the plates. Since the volume of the dielectric is αl, the energy w stored per unit volume of the dielectric is—

$$w = \tfrac{1}{2}pF\frac{E}{l} = \tfrac{1}{2}pF^2 \qquad (16a)$$

29. Electrostatic Flux Density or Electric Displacement: [**Definitions**]—The quantity pF which appears in the expression for the energy stored in unit volume of the dielectric (equation 16a), is termed the **electric displacement** or the **electrostatic flux density**, and is symbolized by $\mathbf{D}$.

$$\mathbf{D} = p\mathbf{F} \qquad (17)$$

[16] Equation (13) has apparently been deduced by analysis, but the analysis is based upon the assumption that the fields of elementary charges may be superposed in the manner stated in Section 15. At this point in the development of these relations, this principle is to be regarded as an assumption not fully demonstrated. The confirmation by experiment of the relation expressed in equation (13) between Q and E and the cross section α and thickness l of the dielectric, is a partial justification of the principle. Equations (13) and (14) are, therefore, to be regarded as experimentally determined relations rather than as deductions.

The expression (16a) for the energy stored in unit volume of the dielectric may now be written in the form—

$$w = \tfrac{1}{2}FD \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

The electrostatic flux density **D** at any point in the dielectric is to be regarded as a vector quantity which in conjunction with the vector quantity **F**—the electric intensity at the point — serves to specify the state of the medium. In the expression for the energy stored per unit volume, if electric intensity **F** is regarded as analogous to stress in a mechanical system, then the electrostatic flux density **D** is analogous to strain.

The quantity pF, which has been termed the electrostatic flux density, has been arrived at from an experimentally determined expression for the capacity of a parallel plate condenser. The capacity in turn was defined as a proportionality constant between the charge in the condenser and the difference of potential between the plates. By retracing these relations, the unit electrostatic flux density may be defined in terms of the quantity of electricity per unit area of the parallel plate condenser. Thus, in the case of the parallel plate condenser, by definition,

$$D \text{ or } pF = p\,\frac{E}{l}$$

But, from equation (11),

$$p\,\frac{E}{l} = \frac{Q}{\alpha}$$

Therefore

$$D\left(= \frac{Q}{\alpha}\right) = \frac{dQ}{d\alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18a)$$

Equation (18a) furnishes the following definition for unit electrostatic flux density:

Unit electrostatic flux density is the electrostatic flux density which results in the medium of a parallel plate condenser when the plates are charged with unit quantity of electricity (one coulomb) per unit area. The unit of electro-

static flux density may, therefore, be termed the **coulomb per sq. cm.**

$$\text{The practical coulomb per sq. cm.} = \begin{cases} 3\times10^{9}\ \text{E. S. coulombs per sq. cm.} \\ 10^{-1}\ \text{E. M. coulombs per sq. cm.} \end{cases}$$

The quantity of electricity per unit area, $\frac{dQ}{d\alpha}$, is symbolized by σ_c, and is termed the surface density of charge.

$$\sigma_c = \frac{dQ}{d\alpha} \quad \text{(18d)}$$

30. Electrostatic Flux: [**Definition**]—The quantity $pF\alpha$ occurring in the expression for the energy stored in the dielectric of the parallel plate condenser is termed the **electrostatic flux** over the cross-sectional area α of the dielectric, and is symbolized by Ψ'.

$$\Psi' = pF\alpha = D\alpha \quad \text{(18b)}$$

The expression (15a), for the energy stored in the dielectric between the parallel plates may now be written in the form

$$W = \tfrac{1}{2}\Psi' E \quad \text{(15)}$$

By the electrostatic flux ($d\Psi'$) over any small element of surface ($d\alpha$) at a point in the field where the electrostatic flux density has the value (D) is meant the product of the area ($d\alpha$) times the component of the electrostatic flux density normal to the surface at the point. The direction in which the electrostatic flux density vector points is taken as the direction of the electrostatic flux across the surface.

$$d\Psi' = D\cos\gamma\, d\alpha \quad \text{(18c)}$$

in which, γ is the angle between the vector representing the electrostatic flux density and the normal to the surface at the point.

By the electrostatic flux Ψ' over an extended surface is meant the sum of the fluxes over all the elementary areas of which the extended surface is composed. That is, the electrostatic flux is the integral taken over the surface of the

normal component of the electrostatic flux density times the differential area (dα).

$$\Psi = \int D \cos \gamma \, d\alpha \qquad (18)$$

Since the unit of electrostatic flux density is termed the coulomb per sq. cm., and electrostatic flux is the product of displacement times area, the **unit electrostatic flux** may be termed the coulomb.

The practical unit of electrostatic flux (coulomb) equals $\begin{cases} 3 \times 10^{11} \text{ E. S. units of flux} \\ 10^{-1} \text{ E. M. units of flux} \end{cases}$

31. Gauss's Theorem, or the Divergence of the Electric Displacement: [Generalization][17]—Gauss's theorem may be thus stated: The surface integral of the electrostatic flux density outward if taken over any closed surface (or the electrostatic flux outward) is equal to the quantity of electricity Q enclosed within the surface.

$$\int^{\text{closed surface}} D \cos \gamma \, d\alpha = Q \qquad (19)$$

Suppose both members of equation (19) are divided by the volume (v) enclosed by the surface, and that this volume is made infinitesimally small. The right member of the equation, dQ/dv becomes the volume density of electricity ρ at the point P enclosed by the surface. The left member is a quotient obtained by dividing the surface integral of the outward electrostatic flux density $\mathbf{D}$ taken over an infinitesimal surface by the volume enclosed by the surface. This quotient is termed the divergence of the vector $\mathbf{D}$ at the point P, and is written div $\mathbf{D}$. Whence, Gauss's theorem when applied to infinitesimal volumes is written in the form

$$\text{div } \mathbf{D} = \rho \qquad (19a)$$

$$\text{in which } \rho = \frac{dQ}{dv} \qquad (20)$$

[17] For electric charges in an infinitely extended homogeneous medium, Gauss's theorem may be deduced by the aid of Coulomb's law of force and the principle of the superposition of electric fields.

32. Permittance and Elastance of a Dielectric Circuit in Terms of Its Dimensions: [**Exp. Det. Rel.**]—Suppose any portion (filament or slice) of a dielectric is so selected that the electrostatic flux over all cross sections has the same value. The ratio of the electrostatic flux Ψ' over a cross section to the potential difference E between the ends of the filament is termed the **permittance** C of this portion of the dielectric.

$$C = \frac{\Psi'}{E} \quad \text{.......... (8)}$$

A dielectric circuit has unit permittance if a difference of potential of one volt gives rise to a flux of one coulomb. Such a circuit may be said to have a permittance of one **coulomb per volt,** or one **farad.** From equation (14), the permittance C between the parallel plane ends of a right cylinder of dielectric of length l and cross-sectional area α is equal to the permittivity p of the dielectric times the cross-sectional area α divided by the length l.

$$C \text{ (of a cylinder)} = p\,\frac{\alpha}{l} \quad \text{.......... (14)}$$

The ratio of the difference of potential E to the electrostatic flux Ψ' is termed the **elastance** of the circuit.

$$S = \frac{E}{\Psi'} \quad \text{.......... (8a)}$$

The unit of elastance may be termed the **volt per coulomb** or the **daraf.** The elastance S of a cylinder of material is equal to its length times the reciprocal of the permittivity (termed the elastivity σ) divided by the cross section.

$$S \text{ (of a cylinder)} = \sigma\,\frac{l}{A} \quad \text{.......... (14a)}$$

33. Permittivity and Elastivity: [**Definitions**]—(Also see Section 14.) The results attained by the introduction and use of the permittivity (p) and by discarding the dielectric constant k may now be seen. If in formulae 13, 14, 15, 16a, and 17, p is replaced by its value in terms of (k),

namely $k/4\pi$, these formulae will all contain the factor $1/4\pi$.

From equation (17), the permittivity may be regarded as a proportionality constant expressing the ratio of the electrostatic flux density **D** to the electric intensity **F** in the medium. Since displacement is expressed in coulombs per sq. cm., and intensity in volts per cm., the unit of permittivity may be termed the **coulomb per sq. cm. per volt per cm.**, or the **coulomb-volt-cm.**, or the **farad-cm.** A material is of unit permittivity if a potential difference of one volt between opposite faces of a centimeter cube of the material is accompanied by an electrostatic flux of one coulomb per sq. cm.

By the **elasticity** σ of a material is meant the reciprocal of its permittivity. The unit may be termed the **volt per cm. per coulomb per sq. cm.** or the **volt-coulomb-cm.**, or the **daraf-cm.**

The permittivity of free space is $\begin{cases} .0796 \text{ E. S. farad-cms.} \\ 8.842 \times 10^{-23} \text{ E. M. farad-cms.} \\ 8.842 \times 10^{-14} \text{ practical farad-cms.} \end{cases}$

34. Dielectric Strength or Disruptive Gradient: [**Definition**]—If, by increasing the potential between two electrodes, an attempt is made to increase the potential gradient without limit in a dielectric, a gradient is eventually reached at some point in the dielectric at which the dielectric "fails" as an insulator. This failure or break down of the insulating medium is indicated by the formation of a brush discharge if the gradient is high in only a limited portion of the distance between the electrodes, and by the passage of a spark if the gradient is uniformly high. The gradient at which this breakdown of the medium occurs is called the **disruptive gradient** or **dielectric strength** of the medium. Dielectric strength is expressed in **volts per cm.**

35. The Electric Convection, Conduction, and Displacement Currents: [**Definitions**]—The continued passage of electric charge across any surface is termed an **electric current** across the surface.

If the charge is carried across the surface by charged moving matter (for example, by pith balls, dust particles, atoms, molecules, endless belts, or rotating disks), the current is termed a **convection current** (symbolized by I_{cv}).

If the charge passes or flows through a conductor, the current is termed a **conduction current** (symbolized by I_c).

The strength of the convection or conduction current (or, briefly, the current) across any surface is defined as equal to the net quantity of electricity which by convection or conduction crosses the surface in unit time, or as equal to the net rate at which electric charge crosses the surface. The **direction of the current** is defined to be the direction in which the positive charge crosses the surface, or the direction opposite to that in which the negative charge crosses it. If in an interval of time (dt), (dq_1) represents the net +charge crossing the surface in the direction AB, and (dq_2) represents the net −charge crossing in the opposite direction, the net quantity (dq) crossing the surface is (dq) = (dq_1) + (dq_2), and the current I in the direction AB is—

$$I_c = \frac{dq}{dt} \qquad (21)$$

Unit current (termed the **ampere** in the practical system) is the current which results in the passage across the surface of unit quantity of electricity (one coulomb) per second.

$$\text{The practical ampere equals} \begin{cases} 3 \times 10^9 \text{ E. S. amperes} \\ 10^{-1} \text{ E. M. amperes} \end{cases}$$

The **rate of increase** of the electrostatic flux Ψ across any surface is termed the electric **displacement current** I_f across the surface. The unit is the ampere—a rate of increase in the electrostatic flux across the surface of one coulomb per second.

$$I_f = \frac{d\Psi}{dt} \qquad (21a)$$

By **the** current I across any surface is meant the sum of the convection, conduction, and displacement currents.

$$I = I_{cv} + I_c + I_f = \frac{dq_{cv}}{dt} + \frac{dq_c}{dt} + \frac{d\Psi}{dt} \qquad (21b)$$

36. Current Density: [**Definition**]—By the current density $\mathbf{I}_d$ at any point in a given plane surface, is meant the current crossing the plane per unit area at the point. By **the** current density at any point in a medium is meant the maximum current density at the point; that is, the current density measured in a plane normal to the direction of flow of current at the point. The current density at a point is a vector quantity pointing in the direction of current flow. (See Section 50.)

$$\mathbf{I}_d = \frac{d\,I}{d\alpha} \qquad (22)$$

37. Kirchhoff's First Law: [**Deduction**]—The following law follows from equation (19) and from the manner in which current has been defined. "At every instant of time, the current crossing any closed surface is zero," or "At every instant of time, the surface integral of the current density outward if taken over any closed surface is zero."

$$\text{I (across a closed surface) or} \int^{\text{closed surface}} I_d \cos\gamma \, d\alpha = 0 \qquad (23)$$

When applied to an infinitesimal volume, this law may be stated thus: "The divergence of the current density is zero."

$$\operatorname{div} \mathbf{I}_d = 0 \qquad (24)$$

Since the current entering any closed surface at one point must leave it at some other, current does not originate at any point. All current filaments are closed loops returning into themselves.

For the special case in which the only currents to be considered are conduction currents flowing in a network of conductors, the law may be stated in the special form given by Kirchhoff, namely, "At every instant of time, the algebraic sum of the currents flowing **toward** any common junction point is zero."

$$\Sigma I_c = 0 \qquad (23a)$$

38. Current Manifestations: [**Exp. Det. Rel.**]—An electric current may manifest itself in one or more of the following ways:

1. It may result in a separation (or a recombination) of electric charge. This is the effect which is utilized in the E. S. system of units, and in this digest, to define the unit current.

2. If the current passes through an electrolyte, simpler component parts of the electrolyte are deposited or set free at the electrodes. This is the effect which is utilized in the practical system of units to define the unit current.

3. A current flowing in a conductor exerts a force on a magnetic pole tending to cause the pole to rotate about the conductor. This effect is utilized in the E. M. system of units to define the unit current.[18]

4. If a current passes through a conductor, the conductor is heated thereby.

5. Mechanical forces—attractive and repulsive—exist between different portions of a conductor carrying a current, and between different conductors carrying different currents.

6. An electromotive force is induced in the circuit and in other circuits if the current in a circuit varies, or if either

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[18] The definitions leading up to the definition of unit current in the E. M. system of units are as follows:

39. Coulomb's Law of Force between Magnetic Poles: [**Exp. Det. Rel.** (**1785**)]—The force of repulsion f between two concentrated magnetic poles in an infinitely extended homogeneous medium is directly proportional to the product of the pole strengths and inversely proportional to the square of the distance l between the poles, and it depends upon the medium by which the poles are surrounded.

$$f = \frac{s_1 s_2}{\mu l^2} \qquad \ldots\ldots (25)$$

40. Permeability: [**Definition**]—The constant μ appearing in the denominator of the expression for the force between two magnetic poles is termed the **permeability** of the medium. In the E. M. system of units, the permeability of free space is taken as unity.

41. Unit Magnetic Pole: [**Definition**]—A magnetic pole is of unit strength if, when placed in a vacuum at unit distance from an equal pole, the force of repulsion between the two will be one dyne.

42. Magnetic Intensity, or Magnetic Force: [**Definition**]—The magnetic intensity at any point in a magnetic field is defined as a vector $\mathbf{H}$ whose magnitude is equal to the force in dynes with which a unit north-seeking pole placed at that point would be acted upon, and whose direction indicates the direction of the force upon the pole.

43. Electric Current: [**Definition**]—The **current** flowing in a conductor is defined as a quantity whose value is directly proportional to the force with which a given magnetic pole is acted upon when placed at a given point in the magnetic field of the conductor. **Unit current** is defined as that current which, flowing in a long straight wire having a remote return, acts with a force of two dynes upon unit pole placed at unit distance from the center of the wire. It may also be defined as that current which, flowing in a conductor in the form of a circular arc one cm. in length, acts with a force of one dyne on a unit magnetic pole placed at the center of the circle.

of the circuits (carrying currents) is moved relative to the other circuit.

44. Faraday's Laws of Electrolysis: [Exp. Det. Rel. (1853)] Electro-Chemical Equivalent of a Substance: [Definition]—Law 1. The mass (M_a) of an ion (a) (an atomic, or atomic group, carrier of charge) deposited on an electrode, or there dissolved, during the passage of current through an electrolytic conductor, is proportional to the electric charge crossing the electrode during the deposit or solution.

$$M_a = K_a Q \qquad (26)$$

$$= K_a It^{19} \qquad (26a)$$

The proportionality constant K_a is called the **electro-chemical equivalent of the ion.** The electro-chemical equivalent of an ion is the mass of the ion in grams deposited per unit quantity of electricity, or deposited per second by unit current.

Law 2. The electro-chemical equivalent of any ion is directly proportional to its atomic (or combining) weight and inversely proportional to its valence; that is, the electro-chemical equivalent of the substance is proportional to its chemical equivalent.

45. Causes of Differences of Potential: [Exp. Det. Rel.] Electromotive Force and Voltage: [Definitions]—A separation and static distribution of charges brought about by **mechanical means** (for example, by the frictional contact and separation of dissimilar materials) is not the only cause of differences of potential between points of an electric field. A difference of potential may exist between points of a **connected system of conductors** under the following conditions:

1. If electricity is continuously conveyed by mechanical means from one point of the system of conductors and delivered to it at another point, as by an electrostatic generator.

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[19] This relation has been used in the practical system of units in the legal definition of the concrete standard of unit current. This definition is "The International Ampere is the unvarying electric current which, when passed through a solution of nitrate of silver in water, deposits silver at the rate of .00111800 of a gram per second."

2. If the system consists of two dissimilar conductors, as copper and zinc, in contact with each other at some point.

3. If the system contains dissimilar conductors dipping into a solution which reacts chemically with one or both of the conductors.

4. If the system contains dissimilar conductors with the junctions at different temperatures, or if there is a difference in temperature between two points of the same conductor.

5. If the conductors or portions of the conductors are moving in a magnetic field.

6. If the conductors or portions of the conductors are traversed by a varying magnetic field which is set up by moving magnets or other current carrying systems.

Systems of this kind contain regions in which energy in some other form, as mechanical, thermal, or chemical, is converted into the electrical form. These regions are said to be the seats or sources of **electromotive forces**—forces which cause a separation of electric charge, and tend to sustain a flow of electricity from one region to another. If a source of this kind is connected in a circuit, an electromotive force (e. m. f.) is said to be **impressed** in the circuit.

The electromotive force E of a source is defined as equal to the energy which is converted into the electrical form per unit quantity of electricity which passes through the source.

$$E = \frac{W}{Q} \text{ (see equation 4)} \qquad (27)$$

Electromotive force may, therefore, be expressed in terms of the same unit as potential difference, namely, the **volt.** The e. m. f. or **voltage** of a source is one volt if one joule of energy is converted into the electrical form per coulomb of electricity passing through the source.

An equivalent definition of the e. m. f. of a source is to define it as equal to the time rate P at which energy is converted into the electrical form in the source, divided by the current passing through the source.

$$E = \frac{P}{I} \qquad (28)$$

The unit of power in the practical system is the **watt** or **joule per second** ($=10^7$ ergs per sec.). In the E. S. and E. M. systems the unit of power is the erg per sec.

The direction of the e. m. f. of a source is taken as that direction through the source in which positive charge tends to flow. That terminal of the source which is at the higher potential is called the + terminal. The source therefore tends to cause a flow of current in the external circuit from the + to the − terminal.

46. Relation between the Conduction Current and the Impressed Electromotive Force: Ohm's Law: [Exp. Det. Rel. (1827)]—When a constant e. m. f. is applied to the terminals of a conductor kept at a constant temperature, the steady current which flows through the conductor is directly proportional to the applied e. m. f.

$$I = GE \qquad (29)$$

$$E = RI \qquad (29a)$$

The resistance R and the conductance G (reciprocals of each other) are constants whose value depends upon the dimensions and material of the conductor. The impressed voltage E is said to be **expended** or **consumed** in the resistance. In order to permit the formulation of a law with reference to e. m. f.'s analogous to Newton's third law (To every action there is always an equal and contrary reaction), the flow of current through a resistance is said to give rise to a **counter e. m. f. of resistance,** which acts in a direction to oppose the flow of current, and is equal to $(-R\,I)$.

47. Heat Developed in a Conductor by the Passage of Current: Joule's Law: [Exp. Det. Rel. (1841)]—When a current of strength I traverses a conductor of resistance R, the applied e. m. f. E equals RI. From equation (28) the power P delivered by the source of the applied e. m. f. is $P = EI = RI^2$. Joule experimentally established the fact that the energy so delivered is all converted into heat in the body of the conductor. Joule's law may be thus stated: When a current of strength I traverses a conductor of resistance R, the rate P at which electrical

energy is converted into heat energy in the conductor is equal to the square of the current multiplied by the resistance.

$$P = RI^2 \qquad (30)$$

48. Resistance and Conductance: [**Definitions**]—The proportionality constant R in equation (29a), in other words, the ratio of the potential difference E between the terminals of a conductor to the current, is called the **resistance** of the conductor. The **unit** of resistance in the practical system is called the **ohm.**[20] A conductor has a resistance of **one ohm** if a current of **one ampere** flows through the conductor when the potential difference between its terminals is maintained constant at **one volt.**

$$\text{The practical ohm equals} \begin{cases} \dfrac{1}{9 \times 10^{11}} \text{ E. S. ohms} \\ 10^9 \text{ E. M. ohms} \end{cases}$$

The reciprocal of the resistance of a conductor is termed its **conductance** G. The unit of conductance is termed the **mho.** A conductor having a resistance of one ohm has a conductance of one mho.

49. Resistance of a Conductor in Terms of Its Dimensions: [**Exp. Det. Rel.**]—By experiment it has been determined that the resistance R between the bases of a right cylindrical conductor (wire) of length l and of cross-sectional area α. is directly proportional to the length and inversely proportional to the cross section.

$$R \text{ (of a cylinder)} = \rho \frac{l}{\alpha} \qquad (31)$$

The conductance G of the cylinder is given by the expression—

$$G \text{ (of a cylinder)} = \gamma \frac{\alpha}{l} \qquad (31a)$$

[20] In the legal definitions of the practical units, the ohm is a fundamental unit. The concrete standard of unit resistance is thus defined. The **international ohm** is the resistance offered to an unvarying electric current by a column of mercury at the temperature of melting ice, 14.4521 grams in mass, of a constant cross-sectional area and of a length of 106.300 centimeters.

The resistivity ρ and the conductivity γ, appearing in these formulae, are constants of the material of which the conductor is composed.

50. Conductivity and Resistivity: [Definitions]—The **resistivity** of a material is the resistance between opposite faces of a centimeter cube of the material, while the **conductivity** is the conductance between the opposite faces of the cube. The unit of resistivity may be termed the **ohm-cm.,** and of conductivity, the **mho-cm.**

From the preceding definitions it follows that the conduction current density $\mathbf{I}_d$ at any point in a substance is related to the electric intensity $\mathbf{F}$ at the point by the equation—

$$\mathbf{I}_d = \gamma \mathbf{F} \qquad (32)$$

Hence the resistivity and the conductivity may be also regarded as proportionality constants expressing the ratio of the conduction current density $\mathbf{I}_d$ at any point in the medium to the electric intensity $\mathbf{F}$ at the point, or the reciprocal of this ratio.

51. The Magnetic Field Accompanying a Current: [Exp. Det. Rel.]—When an electric current flows in a conductor, mechanical forces are found to exist between different portions of the conductor, and between the conductor and other systems of conductors carrying currents in its vicinity. Forces, which tend to cause the poles of a magnet to rotate in opposite directions about the conductor, also exist between the conductor and any magnetized body in its vicinity. That is to say, a current is always accompanied by a **magnetic field**—a region in which these actions are manifested. It is accordingly convenient to say that the current "gives rise to," or "sets up," or "is the cause of," the magnetic field.[21] The current is said to exert a **magnetomotive force** upon the medium surrounding it.

Faraday's studies in electro-magnetic induction show that energy must be expended in starting a current (setting

[21] The two things—the current and the magnetic field—really exist together, and if it facilitates any argument, either may be regarded as producing (in the sense of necessitating) the other.

electric charges in motion), that is, in setting up a magnetic field. The energy so expended is not dissipated, but it reappears when the current ceases. The energy is, therefore, conceived to be stored in the magnetic field. It may be thought of as stored in an **electro-kinetic** form by reason of "concealed motions" in the medium. It will develop that the magnetic state of the medium at any point may be specified by means of two quantities termed the **magnetic intensity** and the **magnetic induction** or **magnetic flux density.**

52. Lines and Tubes of Force: Ampere's Rule for the Positive Direction Along a Line of Force: [Definitions]—If a curve be drawn in the magnetic field so that at every point of the curve its direction coincides with the direction of the force which would be experienced by a magnetic pole at the point, such a curve is called a **line of (magnetic) force,** or a **line of magnetic intensity.** A **tube of force** is the space enclosed by the surface formed by drawing lines of force through every point of a small closed curve. Lines of force are found to be loops always closed and linked with current filament loops.

If the direction of flow of the current in a wire is reversed, a reversal occurs in the direction in which the magnetic pole tends to rotate about the wire. The positive direction along a line of force is arbitrarily taken as the direction in which a north-seeking pole tends to move. The relation between the positive direction for flow of current in a conductor and the positive direction of the lines of force encircling the conductor is given by Ampere's rule, or preferably by a modified statement such as—"Imagine a portion of the conductor to be grasped in the right hand, with the thumb pointing in the positive direction for flow of current, then the fingers encircling the conductor point in the positive direction along the lines of force." From this it follows that if the mathematical convention stated in the following section is adopted, a positive current (that is, a current flowing in the direction arbitrarily designated as the positive direction around the circuit) will give rise to lines of force which thread through the circuit in the positive direction.

53. Mathematical Convention Defining Positive Directions Around and Through a Circuit: [Definition]—The convention generally adopted may be thus stated: The positive direction around a circuit is defined to bear to the positive direction through the circuit the same relation that the direction of rotation of a right-hand screw bears to its direction of advancement. Another statement of the convention is as follows: If a portion of the circuit be grasped in the right hand with the thumb pointing in the positive direction along the circuit, then the direction in which the fingers point is defined as the positive direction through the circuit. This is called the right-hand screw convention.

54. Superposition of Magnetic Fields: [Exp. Det. Rel.]—With a conductor of given configuration immersed in a given medium, as air, the force exerted upon a given magnetic pole in a given position (or upon a small auxiliary exploring circuit carrying a fixed current) is found to be directly proportional to the strength of the current in the conductor.[22] (The strength of the current is here defined by reference either to the rate of separation of charge effected by the current, or to its electrolytic effect). If the magnetic field is due to the current in two or more circuits, the force exerted on a pole at any point P in the field is found to equal the force calculated by experimentally determining the force at P exerted by each circuit acting alone, and then, by the polygon construction, calculating the resultant of these forces. It follows that for the purpose of calculating its magnetic effect, any circuit may be resolved into elementary circuits, or meshes, by imagining bridging conductors carrying equal currents in opposite directions to be connected across the actual conductors in any manner. The force exerted by the actual circuit will be the same as that of the imaginary network, since the magnetic effect of any bridge conductor is zero.

55. Experimental Basis of the First Law of Circulation—In the case of a long straight conductor of circular

[22] These statements are true only if the field is free from ferromagnetic substances. The ferromagnetic substances exhibit saturation effects.

cross section carrying a current in a homogeneous medium, and with a remote return for the current, the forces tend to cause any pole in the vicinity of the wire to rotate in a circle centered about the wire and lying in a plane normal to the length of the wire. The lines of force in the vicinity of the wire are therefore circles having the current carrying conductor as axis. By observations on the period of vibration of a small magnet suspended near a long straight conductor carrying a current I, Biot and Savart determined that the force f acting on a given magnetic pole is inversely proportional to the distance l of the pole from the center of the wire.

$$f = c\,\frac{I}{l} \qquad (33)$$

in which c is a constant whose value (for a given pole) depends upon the units used in expressing f, I, and l. From this it follows that if a given pole moves once around the wire in the + direction along a line of force, the work W done by the field is

$$W \ \left(\text{or} \int f \cos\theta \, dl\right) = 2\pi c I \qquad (33a)$$

In general, in a magnetic field due to any form of electric circuit, when a north-seeking pole moves over any circuit (path) whatsoever, terminating in its initial position, the forces experienced are such that the work done by the field on the pole is proportional to the current flowing in the positive direction across any surface bounded by the path of the pole. Since all current filaments are closed filaments, the current flowing across a surface bounded by the circuit traversed by the pole is linked with this circuit. Therefore the above relation may be thus expressed:

> The work done by the magnetic field on a north-seeking pole which makes one complete circuit in the field, is proportional to the number of **unit-current-filaments** or **unit-current-turns,** linked in the + direction with the magnetic circuit.

Or finally, since the work done is equal to the line integral of the force around the path, the relation may be thus stated:

> The line integral of the mechanical force on a north-seeking pole taken around any complete circuit in the field is proportional to the unit-current-turns linked in the + direction with the magnetic circuit.

This relation is the basis for the **first law of circuitation.** (See Section 58.)

56. Two Aspects of the State-of-the-Medium—The mechanical force exerted upon a magnetic pole at any point in a magnetic field is conceived to be due to the state of the medium at that point. Now, there are several aspects to the "state of the medium." There is—

1. The aspect primarily observed that by reason of its state, the medium exerts a force upon a magnetized body, or upon a conductor carrying a current.

2. The aspect subsequently discovered that this state is related in a simple way to the current giving rise to the field—the force on the pole being directly proportional to the current.[23]

3. The aspect developed still later in Faraday's laws of electro-magnetic induction: namely, if the state of the medium changes relative to a circuit, an electromotive force is induced in the circuit.

Since the force exerted upon a circuit carrying a current is related to the e. m. f. induced in the circuit through the principle of the conservation of energy, there are but two aspects of the medium to be quantitatively expressed. The first quantity is to be used in calculating the force exerted upon a conductor carrying a current, and in calculating the e. m. f. induced in a circuit; the second quantity is to be used in calculating the first from the known current giving rise to the field. These quantities are termed, respectively, the **magnetic induction** or **magnetic flux density,** and the **magnetic intensity** at the point.

[23] If the definitions are formulated in the order followed in the E. M. system, the mechanical force exerted by a current is proportional to the current as a matter of definition and not as a matter of discovery. See Section 43.

57. Magnetic Intensity or Magnetic Force or Magnetizing Force: [**Definition**]—In conformity with the notion ("phraseology" would be a more exact expression) that the current is the cause of the state of the medium, we may say that the current gives rise to a **magnetic intensity,** or **magnetic force,** or magnetizing force at every point in the medium. The **magnetic intensity** at any point in a field is defined as a vector quantity $\mathbf{H}$ whose magnitude is directly proportional to the mechanical force exerted on a given magnetic pole at the point, and whose direction indicates the direction of the force exerted on a north-seeking pole at the point. Clearly then, since the line integral of the mechanical force upon a given pole taken around any complete circuit (path) in the field is directly proportional to the number of unit-current-turns (or ampere-turns) linked with the magnetic circuit, it follows that the line integral of the magnetic intensity taken around the magnetic circuit is likewise directly proportional to the number of ampere-turns encircled.

If, now, the **unit magnetic intensity**[24] is defined as the magnetic intensity at any point in a circular path **one centimeter in circumference** ($\frac{1}{2\pi}$ cm. in radius) centered about a long straight wire carrying **unit current,** the line integral of magnetic intensity around this circumference is unity. The magnetic intensity at any point in this one centimeter circumference may be expressed as an intensity of one **ampere-turn per cm.;** that is, the unit of magnetic intensity may be termed the **ampere-turn per cm.** Therefore, if the unit of magnetic intensity is so defined, the line integral of magnetic intensity around any closed magnetic circuit is not only **directly proportional to** but is also **numerically equal to** the number of **ampere-turns** linked with the magnetic circuit in the + direction.

[24] Another unit of magnetic intensity is the **gilbert per cm.** or the **gauss** which is defined as the intensity at unit distance from the unit pole of the E. M. system. (See Sections 41 and 42.) The ampere-turn per cm. is a unit $.4\pi$ times as large as the gilbert per cm. At the present time, the latter unit is more generally used than the former. The use of the gilbert per cm. necessitates the introduction of 4π as a factor in the following formulae: 34, 35, 45, 46, 46a, 70 and 71. The use of the ampere-turn per cm. banishes the irrational 4π from these formulae, and accomplishes the results sought in the "Rational system of units." (See Section 14. See also Section 73, on magnetivity.)

The practical ampere-turn per cm. equals $\begin{cases} 3\ 10^9 \text{ E. S. amp-turn per cm.} \\ 10^{-1} \text{ E. M. amp-turn per cm.} \end{cases}$

58. Magnetomotive Force: Magnetic Potential Difference: [Definitions] First Law of Circuitation: [Exp. Det. Rel.]—The line integral of the magnetic intensity around any closed magnetic circuit (path) is termed "the magnetomotive force of the current upon the magnetic circuit," or briefly, the **magnetomotive force (m. m. f.).** The **unit** of magnetomotive force is the m. m. f.—line integral of magnetic intensity—along a circuit linking with one turn of wire carrying unit current. This unit is termed the **ampere-turn.** With the unit so defined, the relations outlined above may be thus summarized:

> The magnetomotive force $\mathfrak{F}$ around any closed circuit is equal to the number of ampere-turns linked in the positive direction with the circuit; or, is equal to the current in amperes crossing in the + direction any surface bounded by the circuit.

This has been termed by Heaviside the **first law of circuitation.** (Circuitation: the operation, or the result, of taking a line integral around a closed circuit.)

$\mathfrak{F}$ (or the line integral of the vector $\mathbf{H}$) = the surface integral of the current density $\mathbf{I}_d$.

$$\mathfrak{F} \text{ (or} \int \text{H} \cos \theta \, \text{dl}) = \int \text{I}_d \cos \gamma \, d\alpha \qquad (34)$$

This law when expressed in vector notation takes the following form: Dividing both members of (34) by the area α of the surface bounded by the magnetic circuit,

$$\frac{\int \text{H} \cos \theta \, \text{dl}}{\alpha} = \frac{\int \text{I}_d \cos \gamma \, d\alpha}{\alpha}.$$

If α becomes an infinitesimal area normal to the direction of current flow at a point P, the right member of this equation is the current density at the point P. The left member is

termed "the curl[25] of the vector $\mathbf{H}$ at the point P." Whence—

$$\text{curl } \mathbf{H} = \mathbf{I}_d \qquad (35)$$

The line integral of the magnetic intensity $\mathbf{H}$ taken along a given path ACB between two points A and B is called the drop in magnetic potential between the points A and B along the path traversed in taking the line integral.

59. Magnetic Intensity at a Point Due to an Elementary Length of a Circuit: Ampere's Law: [Generalization]—The magnetic intensity at a point P due to the current in any electric circuit in an infinitely extended homogeneous medium, may be calculated by the following procedure:

1. Divide the circuit into lengths, each so short that it may be regarded as a straight conductor.

2. Calculate the magnetic intensity ($d\mathbf{H}$) at P contributed by each differential length by the formula—

$$d\mathbf{H} = \frac{I \, dl \sin \theta}{4\pi r^2} \text{ amp-turns per cm} \qquad (36)$$

and assign to the differential vector ($d\mathbf{H}$) the proper direction along a normal to the plane determined by the radius (r) and the length (dl).

3. The resultant (determined by the polygon construction) of all the infinitesimal vectors is the magnetic intensity at P.

60. Vector Potential at a Point: [Definition]—The vector potential ($d\mathbf{A}$) at a point P (Fig. 5) due to the current I in an elementary length (dl) of a circuit, is defined to be a vector drawn from the point P parallel to the length dl in the direction of flow of the current in (dl), and having a magnitude equal to—

[25] The **curl of a vector** $\mathbf{H}$ at any point P and **in any plane** passing through that point is defined as a vector $\mathbf{V}$ whose length is equal to the line integral of the vector $\mathbf{H}$ taken around the boundary of an infinitesimal portion of the plane, divided by the area of the infinitesimal portion. The vector $\mathbf{V}$ is to be drawn normal to the plane, and in the + direction along the normal. At the given point there will be some plane for which this quotient, or curl, has a maximum value. This maximum value is termed "**The** curl of the vector $\mathbf{H}$ at the point P."

$$d\mathbf{A} = \frac{Idl}{4\pi r} \quad \text{.......... (37a)}$$

in which r is the distance from P to the length (dl).

The vector potential **A** at a point P due to the current flowing in the entire circuit, or due to the currents in any

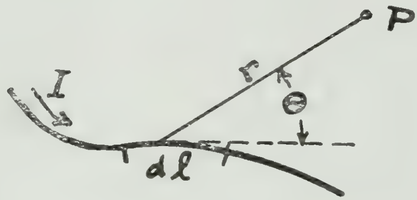

FIG. 5

number of circuits, is the vector through the point which results from the following construction:

1. Dividing the circuit or circuits into lengths each so short that it may be regarded as a straight conductor.
2. Determining the vector potential (d**A**) at P due to each of these differential lengths.
3. Taking the resultant of all the infinitesimal vectors thus determined

$$\mathbf{A} = \Sigma \frac{I\,(dl)}{4\pi r} \quad \text{.......... (37b)}$$

By a comparison of equations (36) and (37b), it may be shown that the magnetic intensity **H** at a point is equal to the curl of the vector potential **A** at the point

$$\mathbf{H} = \text{curl}\ \mathbf{A} \quad \text{.......... (37)}$$

LAWS OF ELECTROMAGNETIC INDUCTION

61. Lenz's Law: [Exp. Det. Rel. (1834)] –The electromotive force which is induced in a body as a result of any variation of the magnetic field with reference to the body, is **in such a direction** that the current which results tends to **prevent** the change which occasions the induced e.m.f.

62. Faraday's Law of Mutual Inductance:[26] **[Exp. Det. Rel.** (1831)]—If the current in a primary circuit varies, an electromotive force e_2 is induced in any secondary circuit; this e.m.f. is directly proportional to the rate of change of the current in the primary circuit, and (from Lenz's law) is in such a direction as to oppose the change of magnetic flux threading the secondary.

$$e_2 = -M\frac{di_1}{dt} \qquad (38)$$

63. Faraday's Law of Self-inductance:[26] **[Exp. Det. Rel.** (1834)]—If the current in a circuit varies, an electromotive force is induced in the circuit which is directly proportional to the rate of change of the current; from Lenz's law this e.m.f. is in such a direction as to oppose the change taking place in the current.

$$e = -L\frac{di}{dt} \qquad (39)$$

Faraday's Law of Motional e.m.f.: [Exp. Det. Rel. (1831)]—See Section 78.

64. Inductance and Mutual Inductance: [Definitions]—The proportionality constant between the voltage induced in a secondary circuit and the rate of change of current in a primary circuit is termed the **mutual inductance** of the two circuits and is symbolized by M. (See Section 66.)

$$M = -\frac{e_2}{\frac{di_1}{dt}} = -\frac{e_1}{\frac{di_2}{dt}} \qquad (38)$$

[26] These statements are very inexact if the field of the conductors contains ferromagnetic materials.

The proportionality constant between the voltage induced in a circuit and the rate of change of the current in the circuit itself is termed the self inductance, or simply the **inductance,** of the circuit, and is symbolized by L.

$$L = -\frac{e}{\frac{d\,i}{d\,t}} \quad \text{(39)}$$

The **unit of inductance** (whether self or mutual) is termed the **henry.** The mutual inductance of two circuits is one henry if a change in the primary current at the rate of one ampere per second induces an e.m.f. of one volt in the secondary. A circuit has a self inductance of one henry if a change in the current at the rate of one ampere per second gives rise to an induced e.m.f. of one volt.

$$\text{The practical henry equals} \begin{cases} \dfrac{1}{9\times 10^{11}} \text{ E. S. henries} \\ 10^9 \text{ E. M. henries} \end{cases}$$

65. Energy Stored in the Magnetic Field of a Single Circuit Carrying a Current: [Deduction]—The energy stored during the establishment of a current in a circuit of inductance L may be thus computed: Assume the resistance of the circuit to be zero, and let i represent the value of the current at any instant.

If the current increases by the amount (di) in the interval of time (dt), the e.m.f. (e) induced in the circuit is $e = -L\,\frac{d\,i}{d\,t}$. During this interval, the source of power must, therefore, deliver a voltage $(-e)$ with the current (i) passing through the source. The energy (dw) supplied by the source of power during the interval (dt) is—

$$dw = (-e)i\ dt = L\frac{di}{dt}\ i\ dt = Li\ di$$

The total energy delivered by the source of power in building up the current from zero to the value I is—

$$W = \int_0^I Li\ di = \tfrac{1}{2}LI^2 \quad \text{(40)}$$

When the current decreases to zero, a voltage is induced in the circuit in the opposite direction, and the same amount of energy is derived from the circuit. The energy $\frac{1}{2}$ LI^2 furnished by the source of power during the establishment of the current I is, therefore, stored and not dissipated. It is said to be stored in the magnetic field: it may be thought of as stored in "concealed" motions established in the field.

66. Energy Stored in the Magnetic Field of Two Circuits Having Mutual Inductance: [Deduction]—Let the two circuits be numbered (1) and (2), and let the following equations express the relations between the induced voltages and the currents:

$$e_2 = -M_1 \frac{di_1}{dt}$$

$$e_1 = -M_2 \frac{di_2}{dt}$$

By an argument in which the two circuits are carried through a complete cycle of changes, the two constants M_1 and M_2 may be shown to be equal. The steps in the argument are as follows:

Assumed conditions: Let both circuits be kept fixed in space, and assume that the resistance of each circuit is zero.

Initial condition: Let the current be zero in each circuit.

Step 1. Let circuit 2 be kept open, and let the current in circuit 1 be brought up to the value I_1. The energy delivered to circuit 1 by its source of power is $\frac{1}{2}$ $L_1I_1^2$.

Step 2: Let circuit 2 be closed and let the current in it be brought up to the value I_2. During this step, let the current in circuit 1 be kept constant at the value I_2 by impressing sufficient voltage in circuit 1 to neutralize the voltage induced in it by reason of the increasing current in circuit 2.

The voltage induced in 1 during the interval (dt) is—

$$e_1 = -M_2 \frac{di_2}{dt}$$

Therefore the energy supplied by the source in circuit 1 during this interval (dt) is—

$$dw = (-e_1)I_1\,dt = M_2\,I_1 di_2$$

The total energy W supplied by the source in (1) while the current (i_2) in circuit 2 is building up to the value I_2 is—

$$W = \int_0^{I_2} M_2 I_1\,di_2 = M_2\,I_1 I_2$$

In addition, the source in circuit 2 has delivered to the circuit the energy $\frac{1}{2}\,L_2\,I_2^2$.

Step 3: Let the current in circuit 1 be decreased to zero, and during this step let the current in circuit 2 be kept constant at the value I_2.

The energy returned by circuit 1 is $\frac{1}{2}\,L_1\,I_1^2$
The energy returned by circuit 2 is $M_1\,I_1\,I_2$

Step 4: Let circuit 1 be kept open, and let the current in circuit 2 be reduced to zero. The energy returned by circuit 2 is $\frac{1}{2}\,L_2\,I_2^2$.

Conclusion: The circuits are now in their initial condition. The energy delivered to the circuits during the cycle is—

$$\tfrac{1}{2}L_1I_1^2 + M_2I_1I_2 + \tfrac{1}{2}L_2I_2^2.$$

The energy returned by the circuits during the cycle is—

$$\tfrac{1}{2}L_1I_1^2 + M_1I_1I_2 + \tfrac{1}{2}L_2I_2^2.$$

If these two quantities are not equal, this cycle, or the reverse of this cycle, will violate the principle of the conservation of energy. Therefore **M_1 must equal M_2**, and the subscripts may be discarded.

The energy stored in the magnetic field of the two circuits is—

$$W = \tfrac{1}{2}L_1I_1^2 + MI_1I_2 + \tfrac{1}{2}L_2I_2^2 \qquad (41)$$

67. Mechanical Force between Two Circuits Carrying Currents: [**Deduction**]—Let the mutual inductance of two circuits carrying the currents I_1 and I_2 be represented

by M. Imagine that one of the circuits is now displaced from its former position by the infinitesimal amount (dx), and imagine that during this displacement the currents are maintained constant at their initial values I_1 and I_2. The displacement is to be a pure displacement without rotation, or without alteration in the form of the circuit. (The distance (dx) is measured in the direction of translation, and is always taken as a positive quantity.)

Imagine the displacement of the circuit to cause no change in the self inductance of the circuit, but to cause an **increase** in the mutual inductance by the amount (dM). The energy stored before the displacement was—

$$\tfrac{1}{2}L_1I_1^2+MI_1I_2+\tfrac{1}{2}L_2I_2^2$$

The energy stored after the displacement is—

$$\tfrac{1}{2}L_1L_1^2+(M+dM)\ I_1I_2+\tfrac{1}{2}L_2I_2^2$$

The increase in the stored energy is I_1I_2 dM.

During the displacement of the circuit, a voltage is induced in each circuit, whose time integral in circuits 1 and 2 is I_2 dM and I_1 dM respectively. Therefore, the energy supplied to each circuit by its source of power in order to maintain the current constant is I_1I_2 dM. The total energy supplied by the sources during the displacement is 2 I_1I_2 dM, while the increase in the stored energy is only I_1I_2 dM, therefore the difference, or I_1I_2 dM, has been expended in doing mechanical work during the displacement of the circuit.

If (f) represents the **component-of-the-force** tending to move the displaced circuit **in the direction of the displacement** (dx) **from the initial to the final position,** the mechanical work done by the moving circuit is f (dx).

$$\text{Therefore } f(dx)=I_1I_2\ dM$$

$$\text{or } f \quad =I_1I_2\frac{d\,M}{dx} \qquad\qquad (42)$$

68. The Magnetic Circuit: Inductance of an Annular Coil in Terms of its Dimensions: [Exp. Det. Rel.]—Energy Stored in Unit Volume of the Field: [Deduction]—It has been determined by experiment that

the inductance of a coil consisting of N turns of wire uniformly distributed about an annular core as in Fig. 6 is given by the expression—

$$L = m \frac{\alpha}{l} N^2 \qquad (43)^{28}$$

in which

α is the cross-sectional area of the core $(= \pi b^2)$

l is the mean circumference of the core $(= 2\pi r)$

and m is a constant whose value depends upon the material constituting the annual core. m is termed the **magnetivity** of the material.[27]

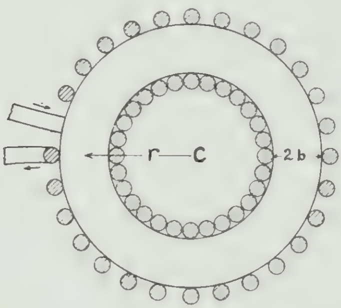

FIG. 6

When a steady current flows in the coil, no magnetic field is found in the space outside the coil. Hence the energy stored in the magnetic field of such a coil is conceived to be stored in the annular core. Within the coil, the lines of magnetic intensity are circles concentric with the axis CC

---

[27] The expression for the inductance is also written in the form—

$$L = 4\pi\mu \frac{\alpha}{l} N^2 \qquad (43a)$$

The constant μ in this expression is termed the **permeability** of the material. (See Section 73.) The magnetivity m of a material is related to its permeability μ by the expression

$$m = 4\pi\mu \qquad (44)$$

[28] These expressions are exact if the circumference l is very great in comparison with the radius b of the cross-sectional area, and if the core consists of free space. They are substantially exact if the core consists of any material save the ferromagnetic materials. For cores of the ferromagnetic materials, the equations are rough approximations for a very limited range of the exciting current.

of the annulus. The magnetomotive force $\mathfrak{F}$ around any of these circles due to the current I in the coil is NI ampere-turns; and from the symmetry, the magnetic intensity H at any point of these circles is approximately $\frac{NI}{l}$ amp-turns per cm.

The energy W stored in the magnetic field (the annular space) is—

$$W \text{ (or } \tfrac{1}{2}LI^2) = \tfrac{1}{2}m\frac{NI}{l}\alpha(NI) = \tfrac{1}{2}(mH)\alpha\mathfrak{F}\ldots\ldots(45a)^{28}$$

Since the volume of the annulus is α l, the energy w stored per cubic cm. is—

$$w = \tfrac{1}{2}mH\frac{\mathfrak{F}}{l} = \tfrac{1}{2}mH^2\ldots\ldots(46a)^{28}$$

69. Magnetic Flux Density or Magnetic Induction: [Definition]—The quantity m**H** occurring in the expression for the energy stored in unit volume of the magnetic field, (equation 46a), is termed the **magnetic induction** or the **magnetic density,** and is symbolized by **B.**

$$\mathbf{B} = m\mathbf{H}\ldots\ldots(47)$$

The expression for the energy stored in unit volume of the magnetic field may now be written in the form—

$$w = \tfrac{1}{2}HB\ldots\ldots(46)$$

The magnetic flux density **B** at any point in a magnetic field is to be regarded as a vector quantity which in conjunction with the vector quantity **H**—the magnetic intensity at the point—serves to specify the state of the medium. In the expression for the energy stored in unit volume, if magnetic intensity is regarded as analogous to velocity in a mechanical system, then magnetic flux density is analogous to momentum.

The quantity mH, which has been termed the magnetic flux density, has been arrived at from an experimentally determined expression, (43), for the inductance of an annular coil. The inductance, in turn, was defined as a proportionality constant between the electromotive force induced

in a coil and the rate of change of current in the coil. By retracing these relations, **unit magnetic flux density** may be defined in terms of the e. m. f. induced in a turn of wire placed in the magnetic field. Thus: By experiment, the inductance of the annular coil of Fig. 6 has the value

$$L = m \frac{\alpha}{l} N^2 \qquad \text{(43)}$$

and by definition—

$$B = mH = m \frac{NI}{l}$$

Whence—

$$L = \frac{B\alpha N}{I} \qquad \text{(48)}$$

Therefore the voltage e induced in the annular coil when the current varies may be written in the form—

$$e\left(= -L\frac{di}{dt}\right) = -\alpha N \frac{dB}{dt} \qquad \text{(49a)}$$

That is to say, the voltage induced in the annular coil is equal to the product formed by multiplying the rate of decrease of the magnetic flux density at any point of the cross-sectional area by the number of turns in the coil and by the cross-sectional area of the annular core. Equation (49a) furnishes the following definition for unit magnetic flux density:

> At any point the magnetic flux density normal to a given plane passing through the point is defined to be changing at **unit rate** in time if the electromotive force thereby induced in a small turn of wire lying in the plane is equal to one volt for every square centimeter of the plane within the boundary formed by the turn of wire.

In other words: **Unit magnetic flux density** is defined to be of such magnitude that an increase in the magnetic flux density at the rate of one unit per second will result in an induced electromotive force of one volt in a turn of wire

which lies in a plane perpendicular to the increment in the flux density, and which includes within its contour unit area of the plane.[29]

By definition—

$$e \text{ or } \int F \cos\theta \, dl = -\int \frac{dB}{dt} \cos\gamma \, d\alpha \qquad (49)$$

In this equation the positive direction for magnetic flux density (through the circuit) is related to the positive direction for voltage (around the circuit) by the right-hand screw convention stated in Section 53.

No short name has been coined for the unit of magnetic flux density in the practical system of units.[30] The name **weber** has, however, been assigned to the product of flux density times surface area. (See the next section.) The unit of magnetic flux density is therefore termed a flux density of one **weber per sq. cm.**

The practical **weber per sq. cm.** = $\frac{1}{300}$ E. S. webers per sq. cm.
10^8 E. M. webers per sq. cm. or
10^8 maxwells per sq. cm.[30]

70. Magnetic Flux and Magnetic Linkages: [Definitions]—The quantity ($mH\alpha$) occurring in the expression (45a) for the energy stored in the annular core is termed the **magnetic flux** over the cross-sectional area α of the core, and is symbolized by Φ.

$$\Phi = mH\alpha = B\alpha \qquad (50a)$$

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[29] This statement—or equation (49)—is to be regarded as more than the definition of **unit** magnetic flux density. It is to be regarded as the fundamental definition of magnetic flux density itself. Equations (43) to (47) do not apply at all exactly to ferromagnetic materials. Magnetic flux density in such materials must be defined as in equation (49).

[30] The unit of magnetic flux density in the E. M. system is known as the **maxwell per sq. cm.** from the fact that the name **maxwell** has been assigned to the product of flux density (expressed in E. M. units) times surface area. The maxwell and the maxwell per sq. cm. are not units of the practical system. (See the footnote to the next section.) The practical unit of flux density is the weber per sq. cm. The micro-weber per sq. cm. is a submultiple of convenient size. Thus, transformer steel is worked at inductions or flux densities as high as 90 to 150 micro-webers per sq. cm.

The expression for the energy stored in the core may now be written in the form—

$$w = \tfrac{1}{2}\mathfrak{F}\,\Phi \qquad (45)$$

By the magnetic flux ($d\Phi$) over any small element of surface ($d\alpha$) at a point in the field where the magnetic flux density has the value (B) is meant the product of the area ($d\alpha$) times the component of the flux density normal to the surface at the point—

$$d\Phi = B\cos\gamma\, d\alpha \qquad (50b)$$

in which, γ is the angle between the vector representing the flux density and the normal to the surface at the point.

By the magnetic flux over an extended surface is meant the sum of the fluxes over all the elementary areas of which the extended surface is composed. That is, the magnetic flux is the integral, taken over the surface, of the normal component of the magnetic flux density times the differential area ($d\alpha$).

$$\Phi = \int B\cos\gamma\, d\alpha \qquad (50)$$

The direction of the vector representing the magnetic flux density **B** is taken as the direction of the magnetic flux across a surface.

Unit magnetic flux is the flux (or surface integral) which results from unit magnetic flux density normal to one sq. cm. of surface. From the following consideration of the connection between the magnetic flux over any unclosed surface and the e.m.f. induced in a conductor forming the boundary or contour of the surface, the unit magnetic flux may be defined in terms of the e.m.f. induced in a turn of wire. Let AC in Figure 7 represent any closed conducting loop forming the boundary of any unclosed surface (diaphragm or cap). Imagine the surface to be divided up into elementary areas or meshes by the network indicated. Let the e.m.f.'s induced around the loop AC and around the boundaries of the individual meshes be all measured in the same direction, for example, in the clock-wise direction.

Then the electromotive force around the loop AC is equal to the sum of the electromotive forces around the

individual meshes of the surface.[31] But, from the definitions of magnetic flux density (equation 49) and of magnetic flux (equation 50), the induced electro-motive force around the contour of any mesh is equal to the rate of decrease of the magnetic flux taken over the surface of the mesh. Therefore the voltage induced in the loop AC will be equal to the

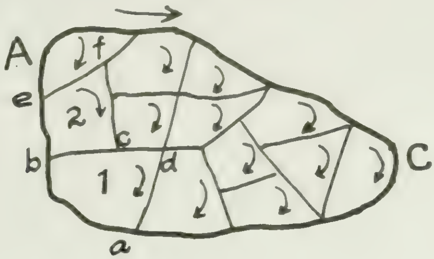

FIG. 7

rate of decrease of the magnetic flux taken over any surface bounded by the loop AC.

$$\int F \cos\theta \, dl \text{ (or e)} = -\frac{d\Phi}{dt} \qquad (51)$$

In this equation, the + direction for electric intensity (around the loop) is related to the + direction for magnetic induction and flux (through the loop) by the right hand screw convention.

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[31] This may be seen by comparing the expression for the e.m.f. around the contour of two adjoining meshes with the expression for the sum of the two e. m. f.'s around the individual meshes. Thus:

The e. m. f. around the contour of meshes 1 and 2 combined is the sum of the e. m. f.'s in the sides—

ab + be + ef + fc + cd + da .... a

The sum of the two e. m. f.'s around the individual meshes 1 and 2 is the sum of the e. m. f.'s in the sides—

(ab + **bc** + cd + da) + (be + ef + fc + **cb**) .... b

In expression b, the side bc common to the adjoining meshes has been traversed twice, but in opposite directions. Therefore, the e. m. f. in a side common to two meshes cancels out in the expression for the sum of the two e. m. f.'s, and the sum of the e. m. f.'s represented by b is equal to that represented by a. By summing up the e. m. f.'s around all the meshes in this manner it is found that the e. m. f. in every portion of the contour AC is added once, and the e. m. f. in every side common to two meshes is added twice but in opposite directions. This establishes the proposition that the e. m. f. induced in the loop AC is equal to the sum of the e. m. f.'s induced around the individual meshes.

Equation (51) furnishes the definition for the unit magnetic flux. The unit flux, which in the practical system is termed the **weber,** may be thus defined.

The magnetic flux through a **loop of wire** is changing at the rate of **one weber per second** if the electromotive force thereby induced in the loop is equal to one **volt.**

$$\text{One practical weber equals} \begin{cases} \frac{1}{300} \text{ E. S. webers} \\ 10^8 \text{ E. M. webers or } 10^8 \text{ maxwells}^{32} \end{cases}$$

The magnetic flux over the surface bounded by a coil is defined by equation (51), and the inductance of the coil is defined by equation (39)—

$$e = -\frac{d\Phi}{dt} \quad\text{..........(51)} \quad \text{and} \quad e = -L\frac{di}{dt} \quad\text{........(39)}$$

$$\text{Whence } L\frac{di}{dt} = \frac{d\Phi}{dt} \quad\text{..........(52)}$$

That is, the inductance of a coil is a proportionality constant which expresses the ratio of the rate of increase of the magnetic flux linked with the coil to the rate of increase of the current in the coil. The inductance of a coil is, therefore, equal to the magnetic flux accompanying unit current in the coil.

$$L = \frac{d\Phi}{di} = \frac{\Phi}{I} \quad\text{..........(52a)}$$

If the loop of wire is not in the form of a single turn forming the contour of a simple surface, but is in the form of a coil consisting of many turns, it is customary to term the surface integral of the magnetic flux density over the many folded surface (each fold being bounded by a turn of the

[32] The unit of magnetic flux in the E. M. system of units—the E. M. weber—is termed the **maxwell** or **line.** Since the volt in the practical system is equal to 10^8 E. M. volts, the practical weber is equal to 10^8 E. M. webers, or 10^8 maxwells. Unfortunately, it is the common practice to express flux in **maxwells** in calculations which are otherwise all carried on in practical units. This practice of mixing systems of units gives rise to those hybrid formulae in which the factor 10^{-8} appears. In the writer's estimation, the use of the maxwell as the unit of flux in engineering formulae should be discontinued. Magnetic flux should be expressed in terms of the practical unit—the weber (or the micro-weber).

coil) the number of magnetic linkages with the coil, or briefly, the **magnetic linkages** Λ. In the case of a coil of N turns in which all of the turns bound substantially the same surface (as in Fig. 8) or in which the magnetic flux over the surface bounded by one turn is equal to the flux over the surface bounded by any other turn (as in a long solenoid or in the annular coil of Fig. 6), the total flux Φ_t or the magnetic linkages Λ over the multifolded surface is approximately equal to N times the flux Φ_1 over the surface bounded by a single turn. The electromotive

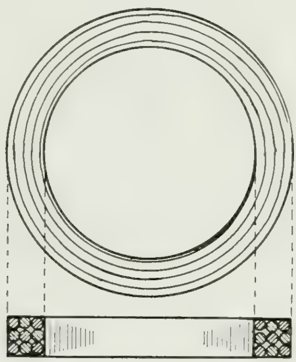

Fig. 8

force e induced in the coil is approximately equal to N times the e.m.f. induced in a single turn.

$$\Lambda \text{ (or } \Phi_t) = N\,\Phi_1 \qquad (53)$$

$$e = -\frac{d\Lambda}{dt} = -N\frac{d\Phi_1}{dt} \qquad (54)$$

71. Continuity of the Magnetic Induction—Let Fig. 9 represent any closed surface. Imagine a closed circuit or loop as ABC to lie on the surface and to divide it into two parts or caps. From the definition of magnetic flux, the rate of decrease of the magnetic flux over either cap is equal to the e.m.f. induced in the loop ABC forming the boundary of the caps. Therefore the rate of decrease of the flux **outward** over one cap must at every instant equal the rate of decrease of the flux **inward** over the

other cap. Or, at every instant the rate of decrease of the flux outward taken over the entire surface must be zero. Whence, the magnetic flux outward taken over any closed surface is zero.

$$\int^{\text{closed surface}} B \cos \gamma \, d\alpha \, (\text{or } \Phi \text{ over a closed surface}) = 0 \ldots . (55a)$$

Suppose both members of equation (55) are divided by the volume (v) enclosed by the surface, and that this volume is made infinitesimally small. The left member is a quotient obtained by dividing the surface integral of the

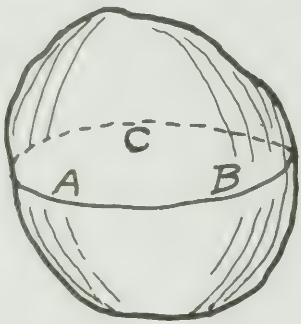

FIG. 9

magnetic induction **B** outward over an infinitesimal surface by the volume enclosed by the surface. This quotient is termed the divergence of the vector **B** at the point P enclosed by the surface.

$$\text{div } \mathbf{B} = 0 \ldots \ldots \ldots \ldots \ldots \ldots (55)$$

72. Magnetance, Permeance, and Reluctance of a Magnetic Circuit in Terms of its Dimension: [Exp. Det. Rel.]—The **magnetance** of a portion of a magnetic circuit, over all cross sections of which the magnetic flux has the same value, is defined as the ratio of the magnetic flux Φ to the exciting magnetomotive force $\mathfrak{F}$.[33]

[33] If the magnetic flux is expressed in maxwells and the magnetomotive force is expressed in gilberts, the ratio of flux to m. m. f. is termed the **permeance** of the magnetic circuit. See note 34.

$$\mathfrak{M} = \frac{\Phi}{\mathfrak{F}} \quad \text{..} \quad (56)$$

A magnetic circuit has unit magnetance if a m.m.f. of **one ampere-turn** gives rise to a magnetic flux of **one weber.** Such a circuit may be said to have a **magnetance** of one **weber per amp-turn.** Since inductance is defined by the equation $L = \frac{N\Phi_1}{I}$ and magnetance by the equation $\mathfrak{M} = \frac{\Phi_1}{NI}$ the inductance of any coil of N turns wound on a magnetic circuit of magnetance $\mathfrak{M}$ will be $L = N^2\mathfrak{M}$. From equation (43), the magnetance between the ends of a right cylinder of material is equal to the magnetivity m of the material times the cross section α divided by the length l of the cylinder.

$$\mathfrak{M} \text{ (of a cylinder)} = m\frac{\alpha}{l} \quad \text{......................................} \quad (57)$$

The ratio of the exciting m.m.f. to the magnetic flux is termed **reluctance** $\mathfrak{R}$ of the magnetic circuit.

$$\mathfrak{R} = \frac{\mathfrak{F}}{\Phi} \quad \text{..} \quad (56a)$$

The unit of reluctance may be termed the **amp-turn per weber.** The reluctance $\mathfrak{R}$ of a cylinder of material is equal to the length of the cylinder times the reciprocal of the magnetivity (termed the reluctivity ν) divided by the cross section.

$$\mathfrak{R} \text{ (of a Cylinder)} = \nu\frac{l}{\alpha} \quad \text{......................................} \quad (57a)$$

73. Magnetivity, Permeability,[34] and Reluctivity: [Definitions] –From equation (47), the magnetivity m of a

[34] The proportionality constant which is at present in general use for expressing the ratio of the magnetic flux density **B** to the magnetic intensity **H** at any point in a body, is termed the **permeability** of the material, and is symbolized by μ. In the definition of permeability, the magnetic flux density is expressed in maxwells per sq. cm., and the magnetic intensity is expressed in gausses or gilberts per cm. The unit may, therefore, be termed the **maxwell-gilbert-cm.** This unit is not a unit of the practical system, but it is an unrationalized E. M. unit which is 4π times as large as the rationalized E. M. unit,—namely, the **E. M. weber amp-turn cm.** Measured by this unit, the permeability of free space

material may be regarded as the proportionality constant which expresses the ratio of the magnetic flux density **B** at any point in the medium to the magnetic intensity **H.** Or, the magnetivity is the magnetance between opposite faces of a centimeter cube of the material. A material is of unit magnetivity if a m.m.f. of one amp-turn between opposite faces of a cm. cube of the material is accompanied by a magnetic flux of one weber. The unit of magnetivity may, therefore, be termed the **weber per sq. cm. per amp-turn per cm.,** or the **weber amp-turn cm.**

The practical **weber amp-turn cm.** equals $\begin{cases} \dfrac{1}{9 \times 10^{11}} \text{ E.S. units} \\ 10^{9} \text{ E. M. units} \end{cases}$

---

in the E. M. system is unity. (See Sections 40 and 68 for definitions of permeability.)

In formulating the rationalized practical units, it is possible to retain the term permeability to designate the ratio of B to H, provided the term is redefined to designate the ratio with magnetic flux density expressed in webers per sq. cm. and with magnetic intensity expressed in ampere-turns per cm. To redefine permeability in this manner, is to apply the name of an old ratio to a new ratio. From thenceforth, the name will have two meanings, and uncertainty and confusion may result. Such uncertainty is to be avoided if possible. It may be readily avoided in one of three ways.

1. The old name when applied to the new ratio may be qualified by other words. Thus

(Old) The permeability of space is unity.

(New) The rationalized practical (r. p.) permeability of space is $4\pi \times 10^{-9}$.

2. The name of the unit in which the ratio is expressed may always be appended to the numerical value of the ratio. Thus

(Old) The permeability of space is 1 maxwell-gilbert-cm.

(New) The permeability of space is $4\pi \times 10^{-9}$ weber amp-turn cms.

3. A new name and a new symbol may be applied to the new ratio. Thus

(Old) The permeability μ of space is unity.

(New) The magnetivity m of space is $4\pi \times 10^{-9}$.

The writer ventures to suggest that the third plan be followed and that the ratio of B to H in rationalized practical units be termed the **magnetivity** of the material. The main considerations to be urged in favor of the third plan, involving the coinage of a new term are:

a. The new term may help us to break with the deplorable practice of attempting to carry on calculations in three systems of units: practical, E. S., and E. M.

b. The second plan, involving the use of the same symbol μ for two ratios, does not work out well for isolated formulae, because the names of the units do not appear in the formulae. For example, consider the uncertainty which will ensue if the formulae (43) and (43a) for the inductance of an annular coil are written—

$$L = \mu \frac{\alpha}{l} N^2 \quad \text{and } L = 4\pi\mu \frac{\alpha}{l} N^2$$

Fessenden has suggested that the ratio of B to H (B in maxwells per sq. cm., and H in E. M. ampere-turns per cm.) be called the **permity** of the material. The term has never come into use. It is therefore available as a designation for the ratio which the writer has ventured to term the **magnetivity.**

The **relative magnetivity**[35] m_r of a substance is defined as the ratio of its magnetivity to the magnetivity of the standard medium—free space. To within one part in a million, the relative magnetivity of air is unity.

$$\text{The magnetivity } m_0 \text{ of free space is} \begin{cases} \dfrac{4\pi}{9 \times 10^{20}} \text{ in E. S. units} \\ 4\pi \text{ in E. M. units} \\ \dfrac{4\pi}{10^9} \text{ in practical units} \end{cases}$$

The **reluctivity** ν of a material is the reciprocal of its magnetivity. The unit may be termed the **amp-turn per cm. per weber per sq. cm.**, or the **amp-turn weber cm.**

74. Intensity of Magnetization: [Definition]—By the **intensity of magnetization** J at any point in a substance is meant the difference between the actual magnetic flux density B at the point and the magnetic flux density which would exist at the point if the substance were replaced by free space **under the same magnetic intensity.** In ferromagnetic materials, the intensity of magnetization is frequently referred to as the metallic induction, or metallic flux density.

$$J = B - m_0 H \qquad (58)$$

There is, apparently, no limit to the values which may be reached by the magnetic flux density B in any material (save the experimental limit which is set by the heating of the conductors carrying the exciting current). On the other hand, no matter how great the exciting magnetic intensity, the intensity of magnetization in any given ferromagnetic material cannot be caused to exceed a certain limiting value. This value is called the saturation value for the intensity of magnetization, or for the metallic induction, in the specified material. For pure iron, the saturation value for the metallic induction is (to within $\pm$ 2 per cent) 211 microwebers per sq. cm.

[35] In magnetivity tables, it will probably be convenient to specify the relative magnetivity of the materials, rather than the magnetivity in absolute units. The relative magnetivity of a material is equal to its relative permeability. Therefore the values tabulated in existing relative permeability tables may be used as the relative magnetivities of the materials.

75. Magnetic Susceptibility: [Definition]—By the magnetic **susceptibility** k of a substance is meant the ratio of the intensity of magnetization J to the magnetic intensity H.

$$k = \frac{J}{H} = \frac{B - m_0 H}{H} = m - m_0 \qquad \ldots (59)$$

It has been shown that the relation between the susceptibility and the magnetic intensity in ferromagnetic materials may be expressed very accurately over quite a range in the value of the magnetic intensity by an empirical expression of the type—

$$k = \frac{1}{a + bH} \qquad \ldots (60)$$

in which a and b are constants to be experimentally determined for each material. Whence, the following empirical relations obtain in ferro-magnetic materials—

$$J\ (= kH) = \frac{H}{a + bH} \qquad \ldots (61)$$

$$B\ (= J + m_0 H) = \frac{H}{a + bH} + m_0 H \qquad \ldots (62)$$

76. Hysteresis Loss in the Magnetic Field—In a ferromagnetic material, the induced magnetic flux density is not directly proportional to the magnetic intensity arising from the current in the exciting winding. Moreover, if the magnetic intensity is repeatedly carried through a cycle of values in such a material, the relation between the corresponding values of the magnetic flux density and magnetic intensity is found to be of the nature shown in Fig. 10. A dissipation of energy occurs when such a cycle is repeatedly traversed. This is indicated by the fact that the temperature of the ferromagnetic material rises above that of the surroundings.

The feature of the cycle is that the magnetic flux density B corresponding to a given value of the magnetic intensity H while H is increasing is less than that value of the magnetic flux density which corresponds to the same value of magnetic intensity but on the decreasing branch of the

cycle. If the values assumed by B and H while the above cycle is traversed are plotted in rectangular coordinates against time, the B curve has the appearance of lagging behind the H curve. From this appearance, this phenomenon has been termed the phenomenon of **magnetic hysteresis** —"hysteresis" being derived from the Greek "to lag

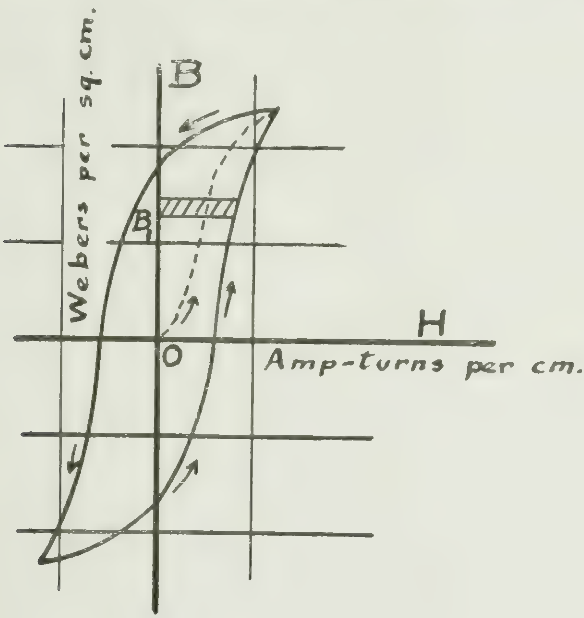

FIG. 10

behind."[36] The cycle of values plotted in Fig. 10 is called a hysteresis loop, and the loss of energy which occurs in the material is called the hysteresis loss.

[36] The expression to the effect that the curve showing the values of the magnetic flux densities lags behind the curve of magnetic intensities is frequently (and erroneously) interpreted to mean that the magnetic induction corresponding to a given value of the magnetic intensity **lags in time behind the intensity, or is attained later in time than the intensity,** and that if the magnetic intensity is carried through the cycle of values with great rapidity the magnetic induction will not be able to follow the variations in the intensity and will be substantially unvarying. This notion is erroneous. The magnetic intensity H and the magnetic flux density B corresponding to it for a given cycle of values occur simultaneously in time. The speed with which the magnetic intensity is carried through the cycle of values apparently has no influence upon the shape and size of the hysteresis loop. (This is at least true up to a frequency of one million cycles per second.)

To compute the energy dissipated in the material when a given hysteresis loop is traversed:

Imagine an annular core (Fig. 6) of length l and cross-sectional area α to be composed of a ferromagnetic material and to be over-wound with an exciting winding of N turns of wire. Let Fig. 10 represent the hysteresis loop which is traversed.

If the magnetic flux density increases from the value B_1 to the value (B_1+dB) in the interval of time (dt), the voltage induced in the exciting winding by the changing induction is—

$$e = \alpha N \frac{dB}{dt}$$

If H_1 is the magnetic intensity corresponding to the induction B_1, the current in the winding must be—

$$i = \frac{H_1 l}{N}$$

The energy (dW) delivered to the exciting winding in the interval of time (dt) is—

$$dW \ (= ei\ dt) = \alpha N \frac{dB}{dt} \frac{H_1 l}{N} = H_1\ dB\ \alpha l$$

If the hysteresis loop is traversed in the interval of time P, the energy delivered to the exciting winding, or expended in the ferromagnetic material, while the loop is being traversed is—

$$W = \int_0^P ei\ dt = \alpha l \int^{\text{one cycle}} H\ dB$$

The energy (w) expended per cubic centimeter of the ferromagnetic material in one cycle is—

$$w = \int^{\text{one cycle}} H\ dB \text{ joules} \qquad \text{(63)}$$

This integral may be seen to equal the area (expressed in webers per sq. cm. amp-turns per cm.) included within the

hysteresis loop. That is, the hysteresis loss in joules per cycle is equal to the area of the hysteresis loop.

77. Empirical Relation between the Hysteresis Loss Per Cycle and the Maximum Value Attained by the Magnetic Flux Density: [**Exp. Det. Rel.**]—From experimental determinations, Steinmetz has derived the empirical expression given in equation 64 for the relation between the hysteresis loss per cycle per cubic cm. of the magnetic material and the maximum value attained by the magnetic flux density in traversing the hysteresis loop. This equation expresses the relation quite accurately (though not rigorously) for quite a range in the value of the magnetic flux density.

$$w = \eta B^{1.6} \text{ joules per cu. cm. per cycle} \quad \text{.............(64)}$$

The exponent (1.6) is an empirically determined constant. The coefficient (η) is called the **hysteresis coefficient** of the material. Its value is determined by experiment for any given material. The values of the hysteresis coefficient (η) for a few materials are as follows:[37]

Material	Hysteresis Coefficient η
Hardened tungsten steel	50 000
Gray cast iron	8 000
Ordinary electrical sheets	2 000
Average silicon steel sheets	530
Best silicon steel sheets	400

78. Faraday's Law of Motional Intensity and Motional Electromotive Force: [**Exp. Det. Rel. (1831)**] —When a body moves relatively to a magnetic field which is itself unvarying when referred to axes fixed with reference to the circuit or magnetized body setting up the field, an **electric intensity** is **induced** in the moving body. The induced electric intensity at any point is normal to the plane determined by the two vectors representing respectively

[37] Steinmetz formula is commonly written in the form—w (in ergs per cu. cm. per cycle) $=\eta_1 B_1$ in which B_1 is expressed in maxwells per sq. cm. The relation between the values of η and η_1 is $\eta = 631000\ \eta_1$.

the velocity of the body at the point (relative to the field) and the magnetic flux density. The induced intensity is in that direction along the normal in which a right-hand screw would advance if rotated in the direction in which the velocity vector must be turned to bring it into parallelism with the B vector. The magnitude of the induced intensity is equal to the product of the magnetic flux density **B** times the component of the velocity normal to the magnetic flux density.

$$F = B \times V \sin \theta \quad \text{.......(65a)}$$

in which, θ represents the angle between the V vector and the B vector.

Or, in vector notation,

$$\mathbf{F} = \mathbf{V} \times \mathbf{B} \text{ (vector product)} \quad \text{.......(65)}$$

The electromotive force induced in a wire moving in a magnetic field is the line integral of the induced electric intensity taken along the length of the wire. The electromotive force E induced in a straight wire of length l moving parallel to itself with the velocity V in a uniform magnetic field is equal to the magnetic flux density B times the component of the velocity perpendicular to the direction of the magnetic flux density times the projection of the length l on a normal to both **V** and **B**. The direction of the induced voltage along the conductor is given by Fleming's right hand rule, namely "Point the first finger of the right hand in the direction of the magnetic flux density, the thumb in the direction of the motion, and the second finger along the wire: the second finger points in the direction of the induced e.m.f." See Fig. 11.

$$E = B \times V \sin \theta \times l \cos \varphi \quad \text{.......(66)}$$

in which θ represents the angle between the **V** and **B** vectors, and φ represents the angle between the normal to these two vectors and the length of the conductor.

This product is equal to the rate at which the conductor sweeps over or cuts across magnetic flux. If the moving conductor constitutes part of a moving circuit, the rate at which this conductor cuts across magnetic flux represents

the rate at which the flux threading the circuit is varying by reason of the motion of this conductor. Therefore, where the flux threading the circuit changes by reason of the motion of the conductor relatively to a steady field, the expression for the induced voltage takes the form—

$$e = -\frac{d\Lambda}{dt} = -N\frac{d\Phi_1}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (54)$$

79. Second Law of Circuitation: [Exp. Det. Rel.]— In all cases in which the flux threading a circuit changes, whether by reason of a change in the current in a neighboring circuit, or by reason of the relative motion of the circuit and a steady magnetic field, or by reason of the motion of a portion of the circuit in the magnetic field set up by the current in the circuit itself, or finally by reason of the variation of the current and magnetic field of the circuit itself, the voltage (e) induced in the circuit is equal to the rate of **decrease** of the number of linkages of the magnetic flux with the circuit. The positive direction for voltage (around the circuit) is related to the positive direction for flux (through the circuit) by the right-hand screw convention stated in Section 53.

$$e = -\frac{d\Lambda}{dt} = -N\frac{d\Phi_1}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (54)$$

This has been called by Heaviside the **second law of circuitation.**

In vector notation, the second law of circuitation takes the following form: In equation (54), let (e) represent the voltage induced around the boundary of a small plane circuit of infinitesimal dimensions.

$$\int F \cos\theta \, dl \ (=e) = -\frac{d\Phi}{dt} \quad \ldots\ldots\ldots\ldots\ldots (51)$$

If both members of this equation are divided by the area enclosed by the boundary around which the line integral of the electric intensity is taken, the right member represents the rate of decrease of the flux density **B** at a point, and by definition (see footnote 25 to Section 58) the left member is the curl of **F**.

Whence $$\text{curl}\ \mathbf{F} = -\frac{d\mathbf{B}}{dt} \qquad (67)$$

80. Mechanical Force Acting on a Conductor Carrying a Current in a Magnetic Field: [Exp. Det. Rel.] —The mechanical force exerted by the field on a short length (dl) of a conductor carrying a current I at a point in the field where the magnetic flux density has the value **B** is—

$$f = B\ I\ dl \sin\theta \qquad (68)$$

in which θ represents the angle between the direction of the length (dl) and the **B** vector at the point. The force is exerted along a normal to the plane determined by the length (dl) and the direction of the magnetic flux density at the point. The direction along this normal in which

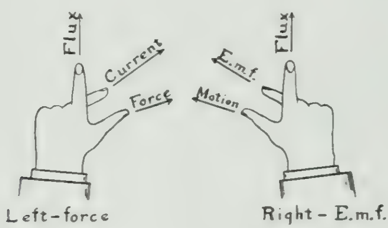

FIG. 11

the force acts may be determined by Fleming's left-hand rule, namely, "Point the first finger of the left hand in the direction of the **B** vector, and the second finger in the direction of the current. The thumb will point along the normal in the direction of the force on the wire." See Fig. 11.

81. Mechanical Force Acting on a Charged Body Moving in a Magnetic Field: [Exp. Det. Rel.]—The mechanical force **f** exerted by the magnetic field on a body carrying the charge Q and moving with the velocity **V** at a point in the field where the magnetic flux density has the value **B**, is equal to the product of the charge times the magnetic flux density times the component of the velocity normal to the direction of the magnetic flux density. The force is normal to the plane determined by V and B and is in

the direction given by Fleming's left-hand rule. In vector notation

$$\mathbf{f} = Q(\mathbf{V} \times \mathbf{B}) \text{ (vector product)} \qquad (69)$$

82. Motional Magnetic Intensity: [Exp. Det. Rel.]—When a body moves relatively to an electrostatic field, **a magnetic intensity** is **induced** in the moving body. The induced magnetic intensity at any point is normal to the plane determined by the two vectors representing respectively the velocity of the body at the point and the electric displacement. The induced intensity is in that direction along the normal in which a right hand screw would advance if rotated in the direction in which the displacement vector must be turned to bring it into parallelism with the velocity vector. The magnitude of the induced intensity is equal to the product of the displacement $\mathbf{D}$ times the component of the velocity normal to the electric displacement. In vector notation

$$\mathbf{H} = \mathbf{D} \times \mathbf{V} \text{ (vector product)} \qquad (70)$$

83. Flow of Energy—Poynting's Theorem: [Deduction]—The energy transmitted by an electric circuit flows or streams through the dielectric surrounding the conductors. The direction of flow at any point in the dielectric is perpendicular to the plane determined by the vectors representing the electric and magnetic intensities at the point. The flow is in that direction along the perpendicular in which a right hand screw would advance if rotated in the direction in which the $\mathbf{F}$ vector must be turned to bring it into parallelism with the $\mathbf{H}$ vector.

The rate P_1 at which energy streams across unit area at the point is given by the expression (Poynting's theorem)

$$P_1 = F \times H \sin \theta \qquad (71a)$$

in which, θ represents the angle between the $\mathbf{F}$ and $\mathbf{H}$ vectors at the point.

Or, in vector notation

$$\mathbf{P}_1 = \mathbf{F} \times \mathbf{H} \text{ (vector product)} \qquad (71)$$

84. "Weber-unit" Magnetic Pole: [Definition] —Let the "weber-unit" magnetic pole—or briefly, the "weber-unit" pole—be defined as a pole from which issues unit magnetic flux (one weber). With the unit poles in the three systems thus defined, the relations stated in Sections 85 and 86 obtain.

From section 79 one practical "weber-unit" pole = $\begin{cases} \frac{1}{300} \text{ E. S. weber-unit poles} \\ 10^8 \text{ E. M. weber-unit poles} \end{cases}$

The unit pole defined in Section 41 (the fundamental definition of the E.M. system of units) is 4π times as great as the E.M. "weber-unit" pole defined in this section. The unit poles defined in this section are called the "weber-unit" poles to distinguish them from the unit poles of the unrationalized systems of units. The unit poles of the unrationalized systems are all 4π times as great as the corresponding "weber-unit" poles.[38]

85. Mechanical Force Acting upon a Pole in a Magnetic Field: [Exp. Det. Rel.]—The mechanical force $\mathbf{f}$ exerted by the field upon a magnetic pole whose strength is s webers, the pole being placed at a point where the magnetic intensity is $\mathbf{H}$ amp-turns per cm., is—

$$\mathbf{f} = s\mathbf{H} \qquad (72)$$

$$= s\frac{\mathbf{B}}{m} \qquad (72a)$$

86. Mechanical Force between Two Poles: [Exp. Det. Rel.]—The force of repulsion f between two concentrated magnetic poles of strengths s_1 and s_2 webers, separated by the distance l in an infinitely extended homogeneous medium of magnetivity m is

$$f = \frac{1}{4\pi} \frac{s_1 s_2}{ml^2} \qquad (73)$$

[38] The "Heaviside rational unit pole" is defined by the equation

$$f = \frac{1}{4\pi} \frac{s_1 s_2}{\mu l^2} \qquad (73a)$$

in which, the value unity is assigned to the permeability μ of space. Therefore, the H. R. E. M unit pole $= \sqrt{4\pi}$ weber-unit poles $= \frac{1}{\sqrt{4\pi}}$ E. M. unit poles as defined in Section 41.

The magnetivity[39] m_o of free space (or air) equals.... (See section 73)

$$\begin{cases} \dfrac{4\pi}{9 \times 10^{20}} \text{ in the E. S. system} \\ 4\pi \text{ in the E. M. system} \\ \dfrac{4\pi}{10^9} \text{ in the practical system} \end{cases}$$

87. Force of Attraction between the Plane Faces of a Ferromagnetic Core Separated by a Short Air Gap: [**Deduction**]—The force of attraction between the parallel plane faces of a long ferromagnetic core which is interrupted by a short air gap is given by the following expression:

$$f \text{ (per sq. cm. of area of the gap)} = \frac{1}{2}\frac{B^2}{m_o} \quad \text{........(74)}$$

in which

B represents the magnetic flux density in the gap

m_o represents the magnetivity of the material filling the gap (usually air).

[39] The magnetivity m_o of free space when expressed in the Heaviside Rational Electromagnetic Units is unity, and when expressed in the Heaviside Rational Electrostatic Units it is $\frac{1}{9 \times 10^{20}}$

TABLE IV

Formulae Expressing the Fundamental Relations between Electrical Quantities

The formulae are tabulated in the sequence in which the relations have been developed in the preceding digest. In an equation of definition, the quantity therein defined is indicated by a sub or a super dot.

Definitions	Experimentally Determined Relations	Deductions and Generalizations
Definition of equal quantities of electricity.		
Selection of units of— 1. length 2. time 3. either mass or force 4. $\left\{\begin{array}{l}\text{permittivity, p, or}\\ \text{quantity of electricity}\end{array}\right.$ $k = 4\pi p$ .... (3) $p_\circ = \left\{\begin{array}{l}0.0796 \text{ in E. S. S.}\\ 8.84 \times 10^{-23} \text{ in E. M. S.}\\ 8.84 \times 10^{-14} \text{ in P. S.}\end{array}\right.$	$f = \dfrac{Q_1\,Q_2}{kl^2}$ .... (1) $f = \dfrac{Q_1\,Q_2}{4\pi pl^2}$ .... (2)	
$\underset{\cdot}{E} = -\dfrac{dA}{dQ}$ .... (4) $\underset{\cdot}{E} = -\int \mathbf{F} \cos\theta \, dl$ (4a)		$E = \dfrac{1}{4\pi p}\left[\dfrac{Q_1}{l_1} + \dfrac{Q_2}{l_2} + \ldots\ldots\right]$ ..(5a)
$\underset{\cdot}{\mathbf{F}} = \dfrac{d\mathbf{f}}{dQ}$ .... (6)		$\mathbf{F} = -\left[\dfrac{dE}{dl}\right]_{max}$ ...... (4c) $\mathbf{f} = Q\mathbf{F}$ .... (6) $\mathbf{F} = \dfrac{Q}{4\pi pl^2}$ .... (7)
$\underset{\cdot}{C} = \dfrac{Q}{E}$ .... (8) $\underset{\cdot}{S} = \dfrac{E}{Q}$ .... (8a) $\underset{\cdot}{S}_m = \dfrac{E_1}{Q_2} = \dfrac{E_2}{Q_1}$ .... (9)	$\dfrac{Q}{E} = C$ (a constant) (8)	$W = \dfrac{1}{2}\,\dfrac{Q^2}{C} = \frac{1}{2}\,SQ^2$ (10a) $= \frac{1}{2}\,CE^2 = \dfrac{1}{2}\,\dfrac{E^2}{S}$ (10b) $= \frac{1}{2}QE$ .... (10c) $W = \frac{1}{2}\,S_1Q_1^2 + S_mQ_1Q_2 + \frac{1}{2}\,S_2Q_2^2$ ...... (11) $f = \frac{1}{2}\,E^2\,\dfrac{dC}{dx}$ .... (12)

TABLE IV—Continued

Definitions	Experimentally Determined Relations	Deductions and Generalizations
	$Q = \frac{p\alpha}{l} E$ ....(13) C (of a cylinder) = $\frac{p\,\alpha}{l}$ (14) S (of a cylinder) = $\frac{\sigma l}{\alpha}$ (14a)	
$\mathbf{D} = p\mathbf{F}$ ....(17) $D = \frac{dQ}{d\alpha}$ ....(18a) $\sigma_c = \frac{dQ}{d\alpha}$ ....(18d) $\Psi' = \int D \cos\gamma\, d\alpha$ (18) $\rho = \frac{dQ}{dv}$ ....(20)		$w = \frac{1}{2} pF^2$ ....(16a) $w = \frac{1}{2} FD$ ....(16) $\int D \cos\gamma\, d\alpha = Q$ (19) div $\mathbf{D} = \rho$..........(19a)
$C = \frac{\Psi'}{E}$ ....(8) $S = \frac{E}{\Psi'}$ ....(8a)		
$I_c = \frac{dq}{dt}$ ....(21) $I_f = \frac{d\Psi'}{dt}$ ....(21a) $I = \frac{dq_{cv}}{dt} + \frac{dq_c}{dt} + \frac{d\Psi'}{dt}$ ....(21b)		
$\mathbf{I}_d = \frac{dI}{d\alpha}$ ....(22)		$\int^{\text{closed surface}} I_d \cos\gamma\, d\alpha = 0$ (23) div $\mathbf{I_d} = 0$..........(24) $\Sigma I_c = 0$..............(23a)

TABLE IV—Continued

Definitions	Experimentally Determined Relations	Deductions and Generalizations
	$M_a = K_aQ$ ....(26) $= K_aIt$ ....(26a)	
$E = \frac{W}{Q}$ ....(27) $E = \frac{P}{I}$ ....(28)		
$R = \frac{E}{I}$ ....(29) $G = \frac{I}{E}$ ....(29a)	$\frac{E}{I} = R$ (a constant) (29)	
	$P = RI^2$ ....(30)	
	R (of a cylinder) $= \rho \frac{l}{\alpha}$ ....(31) G (of a cylinder) $= \gamma \frac{\alpha}{l}$ ....(31a)	$\mathbf{I}_d = \gamma \mathbf{F}$ ....(32)
	$f = c \frac{I}{l}$ ....(33) $W = 2\pi cI$ ....(33a)	
$\mathbf{H}$ varies as $\mathbf{f}$ $\mathfrak{F} = \int H \cos \theta \, dl$ ..(34)	$\int H \cos \theta \, dl = \int I_d \cos \gamma \, d\alpha$ ....(34)	curl $\mathbf{H} = \mathbf{I}_d$ ....(35)
$\mathbf{A} = \Sigma \frac{I(dl)}{4\pi r}$ ....(37b)		$d\mathbf{H} = \frac{Idl \sin \theta}{4\pi r^2}$ ....(36) $\mathbf{H} =$ curl $\mathbf{A}$ ....(37)
$e_2 = -M \frac{di_1}{dt}$ ....(38) $e_1 = -L \frac{di_1}{dt}$ ....(39)	$e_2 =$ (constant) $\times \frac{di_1}{dt}$ (38) $e_1 =$ (constant) $\times \frac{di_1}{dt}$ (39)	

TABLE IV—Continued

Definitions	Experimentally Determined Relations	Deductions and Generalizations
		$W = \frac{1}{2}LI^2$ ....(40) $W = \frac{1}{2}L_1I_1^2 + MI_1I_2 + \frac{1}{2}L_2I_2^2$ ....(41) $f = I_1I_2 \frac{dM}{dx}$ ......(42)
$L = m \frac{\alpha}{l} N^2$ ....(43) $m = 4\pi\mu$ ....(44) $\mathbf{B} = m\mathbf{H}$ ....(47) $\int F \cos\theta \, dl \; (=e) = -\int \frac{dB}{dt} \cos\gamma \, d\alpha$ ..(49)	$L = m \frac{\alpha}{l} N^2$ ....(43)	$w = \frac{1}{2}mH^2$ ....(46a) $w = \frac{1}{2}HB$ ....(46) $L = \frac{B\alpha N}{I}$ ....(48)
$\Phi = \int B \cos\gamma \, d\alpha$ ..(50) e or $\int F \cos\theta \, dl = -\frac{d\Phi}{dt}$ ....(51)		$W = \frac{1}{2}\mathfrak{F}\Phi$ ....(45) $L \frac{di}{dt} = \frac{d\Phi}{dt}$ ....(52) $L = \frac{\Phi}{I}$ ....(52a)
$\Lambda = N\Phi_1$ ....(53)		$e = -\frac{d\Lambda}{dt} = -N \frac{d\Phi_1}{dt}$ ....(54)
		$\int^{\text{closed surface}} B\cos\gamma d\alpha = 0$ (55a) $\text{div}\ \mathbf{B} = 0$ ....(55)
$\mathfrak{M} = \frac{\Phi}{\mathfrak{F}}$ ....(56) $\mathfrak{R} = \frac{\mathfrak{F}}{\Phi}$ ....(56a)	$\mathfrak{M}$ (of a cylinder) $= m \frac{\alpha}{l}$ ....(57) $\mathfrak{R}$ (of a cylinder) $= \nu \frac{l}{\alpha}$ ....(57a)	

TABLE IV—Concluded

Definitions	Experimentally Determined Relations	Deductions and Generalizations
	$m_o = \begin{cases} 1.396 \times 10^{-20} \text{ in E.S.S.} \\ 12.57 \text{ in E.M.S.} \\ 1.257 \times 10^{-8} \text{ in P.S.} \end{cases}$	
$J = B - m_0H$ ....(58) $k = \frac{J}{H} = \frac{B - m_0H}{H} = m - m_0$ ....(59)	$k = \frac{1}{a + bH}$ ....(60)	$J = \frac{H}{a + bH}$ ....(61) $B = \frac{H}{a + bH} + m_oH$ (62)
	$w = \eta B^{1 \cdot 6}$ ....(64)	$w = \int^{\text{one cycle}} HdB$ ....(63)
	$F = V \times B \sin \theta$ ....(65a) $\mathbf{F} = \mathbf{V} \times \mathbf{B}$ (vector product)..............(65) $E = V \times B \sin \theta \times l \cos \varphi$ (66)	
		$\text{curl } \mathbf{F} = -\frac{d\mathbf{B}}{dt}$ ..(67)
	$f = BI\ dl \sin \theta$........(68) $\mathbf{f} = Q(\mathbf{V} \times \mathbf{B})$ (vector product).........(69) $\mathbf{H} = \mathbf{D} \times \mathbf{V}$ (vector product................(70)	
		$P_1 = F \times H \sin \theta$ (71a) $\mathbf{P}_1 = \mathbf{F} \times \mathbf{H}$ (vector product)......(71)
	$\mathbf{f} = s\mathbf{H}$ ....(72) $\mathbf{f} = s \frac{\mathbf{B}}{m}$ ....(72a) $f = \frac{1}{4\pi} \frac{s_1 s_2}{ml^2}$ ....(73) $f = \frac{1}{2} \frac{B^2}{m_0}$ ....(74)	

TABLE V

Circuit Analogies

Analogies between conducting, magnetic, and dielectric circuits, and a metal bar in tension

For the conducting circuit, read line 1
For the magnetic circuit, read line 2
For the dielectric circuit, read line 3
For the metal bar in tension, read line 4

---

1. An electromotive force E (in volts)
2. A magnetomotive force $\mathfrak{F}$ (in ampere-turns)
3. An electromotive force E (in volts)
4. A tension T (in dynes)

is accompanied by—

1. a conduction current I (in amperes)
2. a magnetic flux Φ (in webers)
3. an electrostatic flux Ψ^* (in coulombs)
4. An elongation E (in centimeters)

and the energy—

1. dissipated is at the rate of EI joules per sec.
2. stored in the magnetic circuit is $\frac{1}{2}\,\mathfrak{F}\Phi$ joules
3. stored in the dielectric circuit is $\frac{1}{2}\,E\,\Psi^*$ joules
4. stored in the metal bar is $\frac{1}{2}\,TE$ ergs

The proportionality constants are called—

1. Resistance $R = E/I$ ohms
2. Reluctance $\mathcal{R} = \mathfrak{F}/\Phi$ ampere-turns per weber[40]
3. Elastance $S = E/\Psi^*$ darafs
4. No name $= T/E$ dynes per cm.

or their reciprocals are called—

1. Conductance $G = I/E$ mhos
2. Magnetance $\mathfrak{M} = \Phi/\mathfrak{F}$ webers per amp-turn[40]
3. Permittance $C = \Psi^*/E$ farads
4. No name $= E/T$ centimeters per dyne

When considering a centimeter cube of the material,

1. An electric intensity F (in volts per cm.)
2. A magnetic intensity H (in amp-turns per cm.)
3. An electric intensity F (in volts per cm.)
4. A stress e (in dynes per sq. cm.)

is accompanied by

1. A current density I_d (in amperes per sq. cm.)
2. A magnetic flux density B (in webers per sq. cm.)
3. An electrostatic flux density D (in coulombs per sq. cm.)
4. A strain t (in cms. per cm.)

[40]In the case of a ferromagnetic material, the ratio of the magnetic flux density to the magnetic intensity is not a constant. Hence in such materials, only for ranges in flux density between narrow limits may the reluctance, reluctivity, magnetance, and magnetivity be treated as approximately constant.

and the energy—

1. dissipated per cubic cm. is at the rate of FI_d joules per sec.
2. stored per cubic cm. is $\frac{1}{2}$ HB joules
3. stored per cubic cm. is $\frac{1}{2}$ FD joules
4. stored per cubic cm. is $\frac{1}{2}$ et ergs

The proportionality constants are called—

1. Resistivity $\rho = F/I_d$ ohm (cm.)
2. Reluctivity[40] $\nu = H/B$ amp-turns per cm. per weber per cm.^{2}
3. Elastivity $\sigma = F/D$ volts per cm. per coulomb per cm.^{2}
4. Modulus of elasticity $M = e/t$ dynes per cm.^{2} per cm. per cm.

or their reciprocals are called—

1. Conductivity $\gamma = I_d/F$ mho (cm.)
2. Magnetivity[40] $m = B/H$ webers per cm.^{2} per amp-turn per cm.; Permeability[40] μ
3. Permittivity $p = D/F$ coulombs per cm^2 per volt per cm.
4. No name $= t/e$ cms. per cm. per dyne per sq. cm.

Breakdown of the material—

1. and 2. There are no phenomena in the conducting and magnetic circuits analogous to the failure of the dielectric circuit and of the metal bar.
3. The electric intensity which causes failure of the material is called the Dielectric Strength, and the corresponding flux density is called the Disruptive Flux Density.
4. The stress which causes failure of the material is called the Ultimate Tensile Strength, and the corresponding strain is called the Disruptive Strain.

Constants of cylinders of material of length l and cross-sectional area α

1. Resistance = resistivity x $\frac{l}{\alpha}$ $\quad R = \rho \frac{l}{\alpha}$
2. Reluctance = reluctivity x $\frac{l}{\alpha}$ $\quad \mathcal{R} = \nu \frac{l}{\alpha}$
3. Elastance = elastivity x $\frac{l}{\alpha}$ $\quad S = \sigma \frac{l}{\alpha}$

1. Conductance = conductivity x $\frac{\alpha}{l}$ $\quad G = \gamma \frac{\alpha}{l}$
2. Magnetance = magnetivity x $\frac{\alpha}{l}$ $\quad \mathfrak{M} = m \frac{\alpha}{l}$; Permeance = permeability x $\frac{\alpha}{l}$ $\quad \mathfrak{P} = \mu \frac{\alpha}{l}$
3. Permittance = permittivity x $\frac{\alpha}{l}$ $\quad C = p \frac{\alpha}{l}$

TABLE VI—NOMENCLATURE OF VARIOUS WRITERS

	Intn'l. Elect. Com.	Am. Inst. Elect. Eng.	Faraday	Maxwell	Heaviside	Emtage	Hertz	Kelvin	Lodge	Ewing	Mascart & Joubert	Boltzmann	Thomson & Poynting	This digest
Electric intensity				●		●			●				●	●
Electric force					●		●	●	●		●	●		
Electrostatic field intensity		●												
Electromotive intensity				●										
Potential gradient														
Electric induction						●								
Displacement				●	●				●		●	●		
Electrostatic flux density	●	●												●
Dielectric flux density														
Electric strain													●	
Electric polarization							●							
Electric flux														
Electrostatic flux	●	●												●
Dielectric flux														
No. of lines of induction												●		
No. of lines of force								●					●	
Total displacement				●					●					
Magnetic intensity						●							●	●
Magnetic force				●	●		●	●	●	●	●	●		
Magnetizing force														
Magnetic field intensity		●							●					
Magnetic field	●													
Magnetic induction				●	●	●			●	●			●	
Magnetic flux density	●	●												●
Magnetic polarization							●							
Density of lines of force			●											
Magnetic flux	●	●							●					●
No. of lines of force			●				●	●						
No. of lines of induction				●						●				
No. of magnetic linkages														
Total induction														

APPENDIX A

Examples of the Expression of Magnetic Data in "Ampere-turn Weber" Magnetic Units

The intensity of the earth's magnetic field in the vicinity of Washington is 0.18 amp-turns per cm., and the magnetic flux density is 0.006 micro-webers per sq. cm.

Magnetic fluxes of the following orders of magnitude are used in electrical machines:

Per pole of a 10 kw. 4 pole D. C. motor, 0.01 webers
Per pole of a 6000 kw. 4 pole 25 cycle turbo-alternator, 1.0 webers
In the core of a 20 kw. 60 cycle distributing transformer, 0.01 webers
In the core of a 1500 kw. 25 cycle power transformer, 0.5 webers

---

Sheet steel in commercial power transformers is worked at peak flux densities lying between 90 and 150 micro-webers per sq. cm.

The following empirical relations obtain in ferromagnetic materials for flux densities higher than 60 microwebers per sq. cm.†

$$\left.\begin{matrix}\text{Intensity of magnetization}\\ \text{or}\\ \text{metallic flux density J}\end{matrix}\right\} = \text{Flux density} - \left\{\begin{matrix}\text{Flux density in space}\\ \text{under an equal}\\ \text{magnetic intensity}\end{matrix}\right.$$

$J = B - m_oH$ ....(definition of J)..............(58)

Magnetic susceptibility $k = J/H$........(definition of k)..............(59)

$$k = \frac{1}{a+bH} \quad \text{..(an empirical relation)........(60)}$$

$$\text{Whence, } B\ (=J+m_oH) = \frac{H}{a+bH} + m_oH \text{.................(62)}$$

In "ampere-turn weber" units, a and b have the following values:

For 3.5 per cent silicon steel*	a	b
between H = 0.8 and H = 40 amp-t. per cm...	10600	6230
between H = 40 and H = 160 amp-t. per cm...	73000	5040
For pure nickel........	120000	175000

The values of a and b in the "amp-turn weber" units are $10^9/4\pi$ and 10^8 times as great as their values in the "gilbert maxwell" units.

The relative magnetivity of silicon steel at a flux density of 100 micro-webers is 2830. This is an absolute magnetivity of $2830 \times (4\pi \text{x} 10^{-9})$ or 35.6 microwebers per sq. cm. per amp-turn per cm.

For pure iron, the saturation value J_s for the intensity of magnetization or the metallic flux density is (to within ± 2 per cent) 211 microwebers per sq. cm.

The magnetic field intensities required to set up in iron magnetic flux densities exceeding the saturation value by 10 microwebers or more may be found approximately by dividing the difference between the required induction B and the saturation value J_s by the magnetivity m_o of free space—

$$H = \frac{B - J_s}{m_o}$$

†Kennelly, A. E., *Magnetic Reluctance*, in *Trans. A. I. E. E.*, Vol. 8, 1891, p. 485.

*Ball, J. D., *The Reluctivity of Silicon Steel as a Linear Function of the Magnetizing Force*, in *General Electric Review*, 1913, Vol. 16, p. 750.

APPENDIX B

Common Formulae for Inductance and Capacity Written in "Ampere Ohm Ampere-turn Weber" Units

All dimensions are in centimeters.
Logarithms are to the Naperian base except as noted.
r, r_1 and r_2 represent radii.
l represents length.
α represents area of cross section.
m_w represents the magnetivity of the wire.
m represents the magnetivity of the medium containing the conductor.
p represents the permittivity of the medium containing the conductor.
m_0 (for air) $=1.257\times10^{-8}$.
p_0 (for air) $=8.84\times10^{-14}$.

Capacity—in farads

Two parallel plates.......................... $C=\dfrac{p\alpha}{l}$

(α is the cross-sectional area of the dielectric)
(l is the distance between the plates)

Two concentric spheres.................... $C=4\pi p\,\dfrac{r_1r_2}{r_2-r_1}$

A sphere in space.............................. $C=4\pi pr$
A circular disk in space.................... $C=8pr$

Two coaxial cylinders....................... $C=\dfrac{2\pi pl}{\log\dfrac{r_2}{r_1}}$

Two parallel wires................................ $C=\dfrac{\pi pl}{\log\dfrac{d}{r}}$

(l is the length of one wire)
(d is the distance between wire centers)

A wire parallel to a plane................ $C=\dfrac{2\pi pl}{\log\dfrac{2h}{r}}$

(h is the distance from the wire to the plane)

Inductance—in henries

Two concentric cylinders.................. $L=\dfrac{l}{4\pi}\left(\dfrac{m_w}{2}+2m\log\dfrac{r_2}{r_1}\right)$

Two parallel wires.............................. $L=\dfrac{l}{4\pi}\left(\dfrac{m_w}{2}+2m\log\dfrac{d}{r}\right)$

(l is the total length of the two wires)

A circle of wire.................................. $L=\dfrac{l}{4\pi}\left[\dfrac{m_w}{2}+m\left(2\log\dfrac{l}{r}-3.516\right)\right]$

A square of wire................................ $L=\dfrac{l}{4\pi}\left[\dfrac{m_w}{2}+m\left(2\log\dfrac{l}{r}-4.32\right)\right]$

A long solenoid and
An annular ring...................... $L=\dfrac{m\alpha N^2}{l}$

(l is the length of the solenoid or the circumference of the ring)
(N is the total number of turns)

APPENDIX C

Actions of Electrical Societies Relating to the Magnetic Units*

1889, At the Second International Congress of Electricians meeting at Paris the following proposals were made but were not acted upon:
"Weber" was proposed as the name of a unit equal to 10^8 E M units of magnetic flux.
"Gauss" was proposed as the name of a unit equal to 10^8 E M units of magnetic flux density.

1891, At the Frankfort International Congress the above proposals were again made but no action was taken.

1891, A committee of the American Institute of Electrical Engineers recommended the following *practical* units. No names were assigned to these units and no action was taken by the Institute.
.1 E M unit of magnetomotive force
10^8 E M unit of magnetic flux density
10^8 unit of magnetic flux
10^{-9} unit of magnetic reluctance

1893, The International Electrical Congress at Chicago recommended the use of the unrationalized E M magnetic units. No names were assigned to the units.

1894, The American Institute of Electrical Engineers accepted the recommendation of the Chicago International Electrical Congress and assigned the following names to the units:
"gilbert" for the E M unit of magnetomotive force
"gauss" for the E M unit of flux density
"weber" for the E M unit of magnetic flux
"oersted" for the E M unit of magnetic reluctance

1895, The committee of Electrical Standards of the British Association recommended the tentative adoption of the following units and names:
As the unit of magnetic flux, 10^8 E M units of magnetic flux, to be known as the "weber."
As the unit of magnetomotive force the E M unit m. m. f., to be known as the "gauss."

1900, The Paris Electrical Congress recommended the use of the following names:
"gauss" for the E M unit of magnetic field intensity.
"maxwell" for the E M unit of magnetic flux.

1911, The Standardization rules of the American Institute of Electrical Engineers specify the use of the following units.
For magnetomotive force, the E M unit, to be known as the "gilbert."
For field intensity, the E M unit, to be known as the "gilbert per cm."
For magnetic flux, the E M unit, to be known as the "maxwell."
For magnetic flux density, the E M unit, to be known as the "gauss."

*Abstracted from *Electrical Units and Standards,* Circular No. 60 of the Bureau of Standards.

1914, The standardization rules of the A. I. E. E. add "gauss" as an alternative name for the E M unit of field intensity, but fail to abandon the use of the same name to designate the unit of flux density.

In footnotes to the rules the following statements are made. "The gauss is provisionally accepted for the present as the name of both the unit of field intensity and flux density, on the assumption that permeability is a simple numeric." "An additional unit for m. m. f. is the ampere turn."

Present Status of Names of Units

The names at present sanctioned for the magnetic units, either officially or by usage, are tabulated below.

Quantity	Name of unit			
	E. M. System		Practical System	
	Unrat'z'd	Rationalized	Unrat'z'd	Rational'd
Magnetomotive force	gilbert	E. M. amp-t.	gilbert	amp-turn
Magnetic intensity	gauss, or gilbert per cm.	E. M. amp-t. per cm.	gauss, or gilbert per cm.	amp-turn per cm.
Magnetic flux	maxwell or line	maxwell or line	weber	weber
Magnetic flux density	maxwell per cm.^{2}	maxwell per cm.^{2}	weber per cm.^{2}	weber per cm.^{2}

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TABLE III

ELECTRICAL QUANTITIES, SYMBOLS, AND UNITS

Quantity	Symbol	Defining equation	Name of the practical unit	Dimensional formulae $[U] = [L^a M^b T^c P^d]$				Relative magnitudes of the units		
				a	b	c	d	$\frac{E.M.}{E.S.}$	$\frac{P}{E.S.}$	$\frac{P}{E.M.}$
Length	l	Fundamental	Centimeter	1	0	0	0	1	1	1
Time	t	Fundamental	Second	0	0	1	0	1	1	1

TABLE III

[illegible]

Quantity	Symbol	Defining equation	Name of practical unit	Dimension a	b	c	d	E.M./E.S.	P/E.S.	P/E.M.
Length	l	Fundamental	Centimeter	1	0	0	0	1	1	1
Time	t	Fundamental	Second	0	0	1	0	1	1	1
Mass	m	[illegible]	[illegible]	0	1	0	0	1	10	10
Area	[illegible]	[illegible]	[illegible]	2	0	0	0	1	1	1
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	0	0	1	1	1
Velocity	v	[illegible]	Cm. per second	[illegible]	0	1	0	1	1	1
Acceleration	[illegible]	[illegible]	Cm. per sec. per sec.	1	0	2	0	1	1	1
Force	f	[illegible]	Dyne seven	1	1	2	[illegible]	1	10	10
Work	[illegible]	[illegible]	Joule	[illegible]	1	[illegible]	0	1	10	10
Energy	W	W [illegible]	Joule	2	1	[illegible]	0	1	10	10
Power	P	P [illegible] W	Watt	2	1	[illegible]	0	1	10	10
Dielectric [illegible]	k	[illegible]	[illegible]	0	0	0	1	[illegible]	[illegible]	[illegible]
Permittivity	[illegible]	[illegible]	Coulomb per sq. cm. per volt per cm. or farad per cm.	0	0	0	1	[illegible]	[illegible]	[illegible]
Quantity of electricity	Q	[illegible]	Coulomb	[illegible]	1/2	1	1/2	[illegible]	3 10^9	[illegible]
Electric potential	[illegible]	[illegible]	Volt	1/2	1/2	1	1/2	[illegible]	1/300	10^8
Electric intensity	**F**	**F** [illegible]	Volt per cm.	1/2	1/2	1	1/2	[illegible]	1/300	10^8
[illegible]	[illegible]	[illegible]	[illegible]	1	[illegible]	0	1	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	1	0	0	1	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	Farad	1	0	0	1	[illegible]	[illegible]	10^9
[illegible] displacement density	**D**	**D** **F** [illegible]	[illegible]	1/2	1/2	1	1/2	[illegible]	[illegible]	[illegible]
Surface density of charge	σ	σ [illegible] dQ [illegible]	Coulomb per [illegible]	1/2	1/2	1	1/2	[illegible]	[illegible]	[illegible]
Electric flux	Ψ	Ψ [illegible] D[illegible]	Coulomb	1/2	1/2	1	1/2	[illegible]	3 10^9	[illegible]
Vol. density of charge	ρ	ρ [illegible] dQ [illegible]	Coulomb per cm.³	[illegible]	1/2	1	1/2	[illegible]	3 10^9	[illegible]
Elastivity	[illegible]	[illegible]	Darafarad	0	0	0	1	[illegible]	[illegible]	10^9
Electric current	I	[illegible]	Ampere	1/2	1/2	2	1/2	[illegible]	3 10^9	10^{-1}
Current density	$\mathbf{I}_1$	$\mathbf{I}_1$ [illegible]	Ampere per cm.²	1/2	1/2	2	1/2	[illegible]	3 10^9	10^{-1}
Electromotive force	E	E [illegible] W [illegible] Q	Volt	1/2	1/2	1	1/2	[illegible]	1/300	10^8
Resistance	R	R [illegible] E/I	Ohm	1	0	1	1	[illegible]	$\frac{1}{9\ 10^{11}}$	10^9
[illegible]	[illegible]	[illegible]	[illegible]	1	[illegible]	1	1	[illegible]	[illegible]	[illegible]
Resistivity	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	[illegible]	[illegible]	[illegible]	[illegible]
Conductivity	[illegible]	[illegible]	Mho per cm.	[illegible]	0	1	1	[illegible]	[illegible]	[illegible]
Magnetic intensity	**H**	$\int$ H cos θ dl [illegible] NI	Ampere turn per cm.	1/2	1/2	2	1/2	[illegible]	[illegible]	[illegible]
Magnetomotive force / Magnetic potential	[illegible]	[illegible] $\int$ H cos θ dl [illegible] NI	Ampere turn / Ampere turn	1/2	1/2	-2	1/2	[illegible]	3 10^9	[illegible]
Vector potential	**A**	**A** [illegible] NI	[illegible]	1/2	1	[illegible]	1/2	[illegible]	3 10^9	[illegible]
Mutual inductance	M	M [illegible]	Henry	1	0	[illegible]	1	[illegible]	[illegible]	[illegible]
Inductance	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Magnetic flux density	**B**	**B** $\int$ [illegible]	Weber per cm.²	[illegible]	1/2	0	1/2	[illegible]	1/300	10^8
Magnetic flux	Φ	Φ $\int$ [illegible]	Weber	1/2	1/2	0	$-1/2$	[illegible]	1/300	10^8
Magnetic linkages	Λ	Λ [illegible] ΦN	Weber	1/2	1/2	0	$-1/2$	[illegible]	1/300	10^8
Magnetic [illegible] permeance	$\mathfrak{M}$	$\mathfrak{M}$ [illegible]	Webers per amp. t.	1	0	[illegible]	-1	[illegible]	[illegible]	[illegible]
Reluctance	$\mathfrak{R}$	[illegible] Φ	Amp. t. per Weber	1	0	[illegible]	1	[illegible]	[illegible]	[illegible]
Magnetivity	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	[illegible]	-1	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	2	-1	[illegible]	[illegible]	10^9
Reluctivity	[illegible]	[illegible] B	Ampere turn per cm. per Weber per sq. cm.	2	0	2	1	[illegible]	[illegible]	10^{-4}
Magnetic [illegible]	[illegible]	[illegible]	Weber	1/2	1	0	$-1/2$	[illegible]	[illegible]	10^8
Intensity of magnetization	I	[illegible]	Weber per cm.²	3/2	1	[illegible]	-1	[illegible]	[illegible]	10^8
Susceptibility	[illegible]	[illegible]	[illegible]	-2	[illegible]	2	1	[illegible]	[illegible]	10^9

[illegible]

BULLETIN OF THE UNIVERSITY OF WISCONSIN

No. 888

ENGINEERING SERIES, VOL. 8, NO. 7, PP. 427-490

FUEL CONSERVATION BY THE ECONOMICAL COMBUSTION OF SOFT COAL

BY

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Associate Professor of Steam and Gas Engineering

The University of Wisconsin

MADISON, WISCONSIN

1917

Price 25 cents

BULLETIN OF THE UNIVERSITY OF WISCONSIN

Issued monthly by the University of Wisconsin at Madison, Wisconsin.
Entered as second-class matter July 11, 1916, at the postoffice at Madison, Wisconsin, under the Act of August 24, 1912.

The Bulletin of the University of Wisconsin is published bimonthly at Madison. For postal purposes, all issues in all series of the Bulletin are included in one consecutive numbering as published, a numbering which has no relation whatever to the arrangement in series and volumes.

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BULLETIN OF THE UNIVERSITY OF WISCONSIN

NO. 888

ENGINEERING SERIES, VOL. 8, NO. 7, PP. 427-490

FUEL CONSERVATION BY THE ECONOMICAL COMBUSTION OF SOFT COAL

BY

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MADISON, WISCONSIN

1917

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INTRODUCTION

Dr. H. A. Garfield, Fuel Administrator, in his Fuel Conservation Message, said: "It is the duty of every American to save coal this winter. If every family will save a ton of coal, if every industrial plant will save 10 per cent of the coal it uses, which 10 per cent it now wastes, the coal problem will be largely solved. There is plenty of coal in the ground, but there is a shortage of cars and of labor at the mine. The solution of the coal problem lies largely with the American people. The Government cannot save coal for them; they must save it for themselves. They must not rely wholly upon price fixing, nor upon the already overtaxed transportation system of the country, nor upon the effort to increase production, nor upon the apportionment of coal, nor upon the enforcement of the law. All must cooperate.

"If the householders of the country save one ton out of twelve, they will save ten million tons of coal. They can conserve the coal supply by more economical methods of firing, by sifting ashes, by watching the furnace door, and by heating only the parts of the house in use. To do this is a public duty.

"The opportunity here for business men's organizations throughout the country to cooperate with the state and local fuel administrators is obvious. The patriotic duty of every manufacturer is to consider the problem of scientific firing and see that his firemen are properly instructed."

Many plants waste through unscientific firing and inadequate equipment as much as 50 per cent of the coal they buy. Exceedingly few firemen know the simplest rudiments of combustion. Many who are considered good firemen persist in operating their furnaces so that 20 per cent or more of the fuel is wasted; in order to overcome this they must be given

an opportunity to learn the principles of combustion. The greatest loss found in our power plants takes place in the boiler room, and it is here that the education factor is most neglected. This is especially true of our smaller power plants and heating plants. Our homes and our schools are heated with little or no thought given to economical operation. The writer recently inspected a school in a prosperous community in this state in which a seventy-year old janitor had to fire eight separate furnaces, besides doing the janitor work for all the rooms occupied by six hundred children. This is not an isolated case. Many of our power plants are operating under conditions almost as bad. Under such conditions it is not entirely the fireman's fault if the plant consumes twice as much coal as it should. If the manager or owner of a plant does not appreciate the importance of studying the processes involved, if he does not take into consideration the need and limitations of his employe, if he does not care to make an investment for the proper furnace equipment and instruments, the fireman cannot wholly be blamed for the uneconomical operation of the plant.

Unfortunately, immediate changes to efficient equipment are in many cases impossible just now when our need to save is greatest, but efficient firing and intelligent effort on the part of all power plant owners to do the best they can with the present equipment would mean an enormous saving that would make the coal situation safe instead of critical.

Too much cannot be said in favor of employing engineers and firemen who are intelligent, skillful, and competent in the performance of their duties. These men should be selected on account of their special fitness for their positions, should be thoroughly instructed in their duties, and should be given pay commensurate with their responsibilities and ability. There can be no greater mistake than to permit the waste of the nation's fuel supply and the abuse of valuable public property through the employment of incompetent firemen, just because such persons may be had for low wages.

In large power plants, good engineering literature is kept on file for employees to take home, meetings are arranged between employers and foremen at regular intervals for discus-

sion and instruction on the special problems coming up, new work, and better and more efficient operation. Lectures upon different subjects pertaining to efficient power-plant operation are given, and the men kept in close touch not only with the outside, but also the inside details of their work. In the larger cities, night schools give courses especially adapted to operating engineers and firemen, and the men are encouraged to attend. As a result, they became more interested in their work, more eager for learning, and the plant in which they are employed becomes operated much more economically. In order to create interest and competition, some employers pay their firemen on a "bonus system," where the men's bonus depends on the efficiency with which the plant is run. Such a system induces the men to study means of efficient operation and usually it results in a benefit to employee and employer alike.

It is for the vast number of men who have not the opportunities mentioned above that this bulletin has been written. The suggestions and principles contained herein will, if followed, do much to conserve the nation's fuel supply at this time when such conservation is of vital importance.

ACKNOWLEDGMENTS

This work was prepared at the suggestion of J. G. D. Mack, State Chief Engineer of Wisconsin, and Vice Chairman of the Wisconsin State Council of Defense, and of J. C. White, State Power Plant Engineer of Wisconsin, in order to call statewide attention to the urgent need for fuel conservation and to the means of attaining it. In preparing the bulletin, free recourse has been had to the works of the best authorities on combustion, from which source much subject matter has been used. The writer does not presume to present much that is new, but rather a collection of principles and suggestions in a form that can be widely distributed, and readily followed by those who have charge of the furnaces in our industrial plants, our schools, and our homes.

The author is indebted to H. J. Thorkelson, Business Manager of the University of Wisconsin, for the chapter on the present fuel market; to A. E. Berggren, Assistant Professor of Steam and Gas Engineering, the University of Wisconsin, for the chapter on devices for burning soft coal; to J. C. White, State Power Plant Engineer, for helpful advice and suggestions; and to P. H. Hyland, Department of Machine Design, the University of Wisconsin, for preparing the illustrations.

FUEL CONSERVATION BY THE ECONOMICAL COMBUSTION OF SOFT COAL

THE PRESENT FUEL MARKET

The experience of the European countries indicates that energetic and extraordinary measures must be resorted to by the Federal and State governments in order to deal with the fuel problem during the war.

In Germany the demand for fuel is so far in excess of the supply that only portions of residences are heated, and even the consumption of hot water for household purposes has been limited.

In Great Britain the fuel situation is so serious that no coal is allowed to be exported, and as a result France and Italy have been compelled to secure fuel from the United States.

This new market, together with the additional demands for fuel among the industries and railways of this country, has created an unusual demand, and, in spite of the fact that the coal production for 1916 was over 15 per cent in excess of that of 1915 (an unprecedented year of production), a consumers' panic, in the scramble to secure necessary fuel, particularly for the industries, railways and utilities, brought about abnormal price conditions which emphasized the alarming character of the fuel situation early in the spring.

Large quantities of hard coal normally used in the domestic trade were diverted to the needs of the industries, and the excessive demands for iron ore with consequently high freight rates on the lakes resulted in the large ore steamers moving westward empty instead of loaded with coal as in past years, and causing a depletion at the head of the lakes of the fuel stock for the Northwest.

Canada has drawn an unprecedented amount of coal, amounting to millions of tons in Quebec and Ontario—a coal movement that has recently been checked by the Federal Government.

The alarming prospect for the winter of 1917–1918 was apparent to coal buyers early in the spring of 1917, and, as a re-

sult, large quantities were contracted for by the heavy buyers before the Federal Government intervened. In consequence of these prior contracts, the prices recently fixed by the President have as yet produced very little improvement in the situation. The fuel situation in the entire country is alarming, and just at present is particularly acute in the New England States and in the Northwest.

The *Coal Age* of September 22, 1917, in speaking of the coal market of that week states:

"The City of Boston, for instance, has had to secure coal from a private consumer at a high price in order to keep some of its municipal plants working. At the other extreme of the country on the North Pacific coast, one of the states, unable to get coal for its public institutions, has made arrangements to secure stump acreage and cut firewood therefrom by convict labor."

This brief survey of the fuel situation gives some idea of its acute character and the imperative need of extraordinary measures requiring the active cooperation of all dealers and users of fuel if the situation is to be properly met.

Causes of the Present Situation

The causes of the unusual fuel situation are shown by the preliminary report of the Federal Trade Commission under date of May 19, 1917, which is given in part below:

"1. **Increased Demand**—The marked increase in demand began to be felt about six months ago. In 1916 the shipments of bituminous coal from the mines amounted to about 509,000,000 tons, or about 66,000,000 tons more than in 1915, when previous records in production had been surpassed. Nevertheless, the demand has recently increased to such an extent that not only the unprecedented output of 1916 has been consumed, but stored coal accumulated from the production of previous years has also been practically exhausted. For example, on the docks of Duluth and Superior, there have usually been carried over, at the opening of navigation, some 3,000,000 tons, but this year when navigation opened the docks were practically empty.

"The increased demand has been largely due to industrial expansion and to increased railroad traffic. In many industries plants have run two or even three shifts, while most railroads have transported more freight tonnage than ever before. In some sections of the country, particularly in the Northwest, an exceptionally cold

winter led to the consumption of more bituminous coal than usual for heating purposes.

"2. **Shifting of Markets**—Much of the increase in demand was concentrated in certain localities. There were transportation difficulties due to unprecedented burdens laid by all classes of freight on both rail and water-transportation facilities. All this led to a considerable shift to other sources of supply in the markets usually supplied from certain coal fields. For example, the coal mined in western Pennsylvania, eastern Ohio, and West Virginia did not reach the markets of the Great Lakes and the lower Ohio in the usual quantities. This was because of increased demand from nearer markets, car shortage, decreased lake-transportation facilities, low water for a considerable period on the Ohio River, and floods in the West Virginia mining region. Consequently the Indiana-Illinois fields have been drawn on to supply the deficit. Thus, the tonnage in commercial shipments (coal not for use of railroads) sent to Michigan from the mines of the Illinois-Indiana fields is reported to have been over 17 times as great during the nine months from April to December, 1916, as it was during the entire year preceding.

"One of the important results of this shifting of markets was a buyers' panic, due to the uncertainty of consumers with regard to getting coal from their regular sources of supply. This led in many cases to a frantic bidding of buyers against each other, for that portion of the coal supply (usually known as "spot" or "free" coal) which the mine operators were able to produce and ship in addition to the part of their output sold under contract.

"The proportion of "free" to "contract" coal has varied widely between different fields and mines, but the usual estimates are that from 70 to 90 per cent of the bituminous coal mined is usually sold under contract. Due to the inability of several of the coal fields to adequately supply their usual markets, and to the great increase in the demand for coal, which had not been foreseen and contracted for by the consumers, the prices of "free" coal have risen enormously in all the markets.

"3. **Inadequate Transportation Facilities**—While, as previously pointed out, there is no lack of coal in the ground, or of mines from which it can be obtained, the fact must be borne in mind that few soft coal mines are equipped to store coal. The coal must be loaded into railroad cars as fast as it comes out of the mine. As a general rule, miners do not go into the mine unless the cars necessary to take care of the day's output are on hand at the mouth of the mine.

"Cars enough to carry away the coal as fast as it can be mined are therefore a prime necessity. During the past six months, from a variety of causes, the railroads have not furnished or have not

been able to furnish cars equal to the productive capacity of the mines. Furthermore, where coal has to be carried part of the way by water, there has been difficulty in getting enough boats.

"The principal causes for a lack of adequate rail transportation have been: Car shortage, embargoes on the movement of freight cars, lack of sufficient motive power, and, to some extent, abuse by shippers and consignees of reconsignment and demurrage privileges. Car shortage in some cases appears to have been due to lack of sufficient cars suited to carrying coal; in others, to the diversion to use in other industries of cars generally available for the movement of coal and in others to the much longer hauls required, due to the shift of markets from their normal sources of supply, which required more cars than usual to distribute the same tonnage.

"Lack of terminal facilities adequate to handle the immense volume of freight consigned to certain points resulted in great congestion at those localities. Railroads on whose lines the shipments originated had to place embargoes against shipments consigned to such destinations, until the congestion could be relieved. In some cases there was also a lack of sufficient locomotives to move the coal from the mines to the consumer. Some of the car shortage appears also to have been due to abuses by shippers of reconsignment privileges and demurrage privileges so as to secure a temporary storage of coal in connection with speculative activities to obtain extortionate prices from coal consumers.

"The principal causes for a lack of adequate water transportation have been: Diversion of boats in the coastwise coal trade to other lines of ocean traffic; diversion of boats in the lake coal trade to ocean traffic; and the conflict of the demand for iron ore and grain transportation with the demand for coal transportation.

COAL LEFT ON DOCK AND WATER FREIGHT TAKEN

"This conflict has occurred because of the higher freights paid for eastward-bound iron-ore and grain transportation than for westward-bound coal. The delay incident to loading and unloading coal, and the fact that three of the highly profitable grain or ore cargoes could be transported in the same time in which only two could be carried, if coal were taken westward, has resulted in boats, suitable for carrying coal, going westward empty, using water as ballast instead of coal.

"4. **Labor Conditions**—Statements made at the commission's hearings by mine operators from different coal fields indicate that in some regions there has been and still is an actual shortage of mining labor. The following reasons are generally ascribed: 1. The wages offered in other industries are often higher than those

paid in the mines. 2. The lack of full-time employment in coal mines, due to insufficient car supply, often necessitates shutting down the mines from two to three days in the week, and sometimes running them only part of a working day. Since many of the men are paid on the basis of the tonnage they mine, the resulting enforced idleness cuts down the actual wages received. 3. There has been a considerable emigration of alien miners, who have been called to the colors of the various warring European nations, and there has been little new immigration from any source to fill their places. Some of the Southern mining fields have been seriously hampered by the movement of negro laborers to the North, generally to other industries than coal mining. This migration is probably ended.

"5. **Increased Costs of Production and Distribution**—Considerable information in regard to increases in costs of production and distribution was submitted by coal operators at the commission's hearings, or has been gathered directly by the commission's agents. These increased costs, as far as production is concerned, are comprised chiefly in cost of labor. While there has been a great increase in the prices of supplies, the increase in the cost of supplies per ton of coal has been a much less important factor than is generally claimed. In the distribution there has also been some increase in cost, mainly in increased cost of transportation to the point of consumption.

"The figures already submitted to the commission in regard to costs and to prices at the mine show that most of the present prices now being charged, both on "free" coal and on such few new contracts as the mine operators are entering into, are far in excess of the costs as shown by the operators' books. Many of the operators frankly take the position that they are trying to get for their coal the highest price possible under the present demand, and are refraining, even at prices greatly increased over last year, from contracting their output to the extent of their usual custom.

"They defend this action by claiming that under the operations of the law of supply and demand they have for many years past been getting little more for their coal than the bare cost of production; that the mining of bituminous coal during that period has been a most unprofitable industry; and that this is their chance to recoup themselves for the losses of several years. Accordingly, they are demanding prices at the mine today which run from fifty per cent to several hundred per cent over the cost of their output.

"As a result of this policy, much of the bituminous coal output has been auctioned off to the highest bidders. This has resulted in great profits to certain operators and in special hardship to municipal public utilities, hospitals, and other public and private

charitable institutions, and to domestic consumers, especially in the West and South, where relatively little anthracite is used.

"6. **Lack of Sufficient Storage Facilities**—The most economical way of handling coal is by loading it into cars as it comes from the mine, transporting it to the point of consumption, and delivering it in the same car direct to the retailer or large consumer. At times, there are adequate transportation facilities to keep the coal moving steadily in this way. But the consumption of bituminous coal is largely influenced by seasonal changes, the demand normally being heavy in fall and winter months, and lighter in the spring and summer.

"This leads, in normal years, to the frequent shutting down of mines, because of lack of orders, at a time when there are plenty of coal cars, and, conversely, to traffic difficulties at other times, largely due to inability to secure enough cars to care for the current demands. The obvious remedy for such a state of affairs is the establishment of storage facilities, preferably near the centers of consumption.

"But the relatively great expense of storing coal has prohibited any general establishment of adequate storage facilities. Thus it has happened that many times during the past six months the chief cities of the United States have been faced with a coal shortage which threatened to stop their street cars, cut off their electric light and power and their gas, and shut down the manufacturing industries which support their population.

"7. **Speculative Activities of Some Mine Operators and Brokers**—As already pointed out, from 70 to 80 per cent of the output of the bituminous coal is sold under contract by the mine operators. It is estimated that the railroads of the country consume about one-third of the total production of bituminous coal. Practically all of the railroad supply, in normal times, is under contract. Of the "free" coal produced by the mines, probably about half is sold by them direct to the consumer. The remainder of the "free" coal and a small part of the coal sold under contract reaches the ultimate consumer through the medium of middlemen, such as brokers and retailers.

"Charges of extortionate prices on the part of mine operators and brokers have been brought to the attention of the commission. According to some informants, various mine operators and brokers, through abuses of the reconsignment and demurrage privileges granted by railroads in order to facilitate the regular distribution of coal, have created or increased local shortages and extorted exorbitant prices from the consumers. These charges are now under investigation by the commission."

COAL

Composition of Coal

The fact is almost universally accepted that beds of coal represent accumulations of vegetable matter in varying stages of preservation. These accumulations of vegetable matter, deposited in swampy places or under water, gradually became covered with silt and other material, and during geological ages changed in physical and chemical composition until they finally became coal. Wood fiber (vegetable matter) is the youngest group in the series, while the successive groups, according to depth and age of formation, are known as peat, lignite, bituminous coal, semi-bituminous coal, semi-anthracite, anthracite, and graphite.

The "combustible" part of coal is that part which will burn, and it consists chiefly of "volatile matter" and "fixed carbon." The term "volatile matter" is used to designate the volatile or gaseous matter, exclusive of moisture, which escapes from the coal when it is heated, and which burns with a flame. Solid or "fixed carbon" is carbon in an uncombined state. It forms the coke of the coal, and burns with a glow and without flame. The non-combustible contents of coal are the moisture, the ash, oxygen, and nitrogen. They represent a portion of the weight of the fuel for which no value is received in heat.

Figure 1 shows, in a very general way, the relation of fixed carbon to volatile matter during the transformation of vegetable matter into coal. The horizontal width of the diagram represents the sum of the fixed carbon and the volatile matter. The inclined line divides the horizontal into parts which represent fixed carbon (at the left) and volatile matter (at the right). The progress of transformation is shown at the right of the diagram.

Classification of Fuels

Figure 1 is not exactly correct in that it seems to indicate well-defined divisions between adjacent classes. In reality, the groups blend into each other, and no definite line of divi-

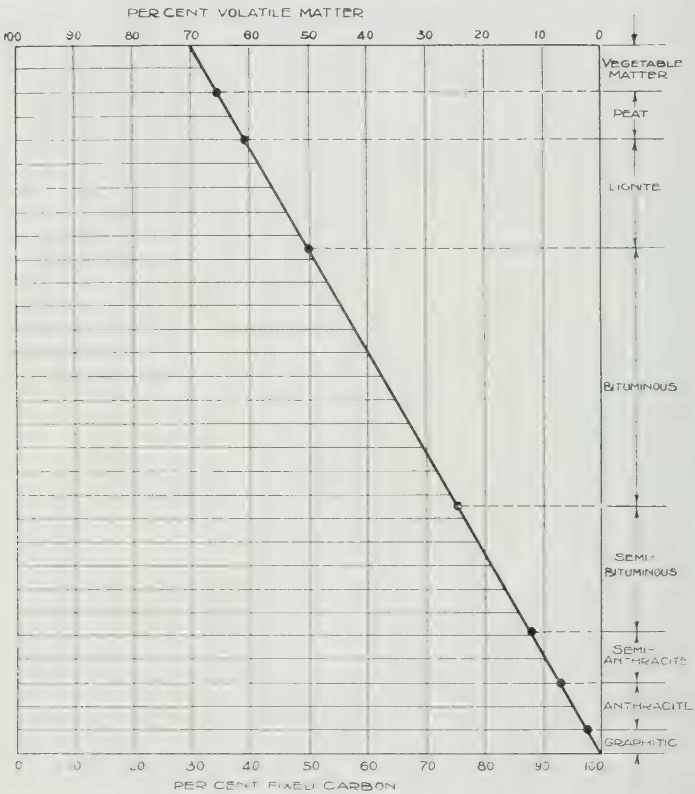

Fig. 1

Relation of Fixed Carbon to Volatile Matter.

sion can be drawn. The diagram is merely for illustration, and should not be used otherwise. Various schemes of classification have been proposed from time to time, but none has been entirely satisfactory, because of the difficulty of adapting a scheme which will apply to lignites as well as to bituminous coal and to anthracite.

A common classification is the following, taken from Kent's *Steam Boiler Economy.*

TABLE I

APPROXIMATE COMPOSITION AND CALORIFIC VALUE OF GENERAL GRADES OF COAL ON BASIS OF COMBUSTIBLE

Kind of Coal	Per Cent of Combustible		B. t. u. per pound of combustible
	Fixed carbon	Volatile matter	
Anthracite	97.0 to 92.5	3.0 to 7.5	14600 to 14800
Semi-anthracite	92.5 to 87.5	7.5 to 12.5	14700 to 15500
Semi-bituminous	87.5 to 75.0	12.5 to 25.0	15500 to 16000
Bituminous—Eastern	75.0 to 60.0	25.0 to 40.0	14800 to 15300
Bituminous—Western	65.0 to 50.0	35.0 to 50.0	13500 to 14800
Lignite	Under 50	Over 50	11000 to 13500

The figures in the above table refer to the combustible portion of the coal. The percentage of ash varies greatly in all the several classes. It may be as low as 5 per cent, and as high as 30 per cent. A more exact classification, and one which appears to apply to all grades of coal, is the carbon-hydrogen-ratio classification proposed by the U. S. Geological Survey.[1]

Characteristics of Fuels

Anthracite, or hard coal, is the name applied to those coals that consist almost entirely of fixed carbon. It contains from 3 per cent to 7 per cent volatile matter, and does not swell when burned. True anthracite is hard, compact, lustrous, and sometimes iridescent, and is characterized by few joints and clefts. In burning, it kindles slowly and with difficulty, is hard to keep alight, and burns with a short, almost colorless flame and without smoke. Nearly all the anthracite used in this country comes from five small fields in Pennsylvania. On account of the limited supply and great demand for domestic purposes, sizes over "pea coal" are prohibitive in price for steam power plant use.

Semi-Anthracite coal has less density, hardness, and luster

[1] *Report of Government Testing Plant,* Professional Paper, No. 98, 1906.

than true anthracite, and can generally be distinguished by its tendency to soil the hands, while pure anthracite will not. It kindles quite readily and burns more freely than the true anthracite. Semi-anthracites are not of great importance in the steam power plant field, on account of the limited supply and high cost.

Semi-bituminous coal is softer than anthracite, contains more volatile matter, kindles more easily, and burns more rapidly. This coal has the highest heat value per pound and ranks among the best steaming coals in the world. The supply is limited, and therefore, because of its high cost, it is not generally used for power purposes. It is quite generally used for domestic heating.

Bituminous, or soft coal, is the most widely distributed and most extensively used coal in steam power plant engineering. It contains a large and varying amount of volatile matter, and produces considerable smoke, unless used in the proper type of furnace and carefully fired.

Bituminous coals are divided into caking, noncaking, and cannel coals. Caking coals swell up, become pasty and fuse together when burning. Non-caking, bituminous coals are the best of the bituminous variety for steaming purposes. They are hard and dense, but somewhat brittle and splintery. They burn freely, do not fuse, and are commonly known as free-burning coals. Cannel coal is very rich in hydrogen and hydrocarbons, ignites readily, and burns with a bright flame. Cannel coal is seldom used as a steam coal, but is used considerably in gas making. It is also used for domestic purposes on account of its cheerful flame when burned in the open fireplace.

Eastern bituminous coals are much more easily pulverized than anthracite, and usually more so than coals of the Illinois type. Repeated handling in transportation, therefore, results in the formation of a large amount of fine material, or slack, which tends to detract from the desirability of these coals.

Lignite is organic matter in the early stages of its conversion into coal. It usually resembles wood in appearance, and is of brownish color. When freshly mined, it contains as high as 50 per cent moisture, and is therefore an unsatisfactory fuel unless this moisture is pressed out before using.

Peat is organic matter in the first stages of its conversion into coal, and is found in bogs and similar places. Its moisture content, when freshly cut, averages from 75 to 80 per cent, and it is therefore unsuitable for fuel until dried. Peat deposits have been found in Wisconsin, and throughout the United States, but it has not been found practicable to use it for steam generating purposes in competition with coal.

Wood is still used as a fuel in certain localities, but the steadily increasing values of even the poorest qualities are rapidly prohibiting its use for steam generating purposes. When first cut, wood contains about 50 per cent of moisture, but after being dried, this is reduced from 10 to 20 per cent. In boiler tests a pound of wood is usually assumed as equal to .4 of a pound of coal. At the present prices it is cheaper, in some localities in northern Wisconsin, to burn wood than hard coal.

Coke is obtained from by-product coke ovens and gas-house retorts. It is manufactured by heating bituminous or soft coal in chambers or retorts into which air is not permitted to enter. In this way, most of the volatile matter is driven off and used as a gas fuel, while the solid carbon of the coal remains as coke. Owing to the increasing difficulty of getting anthracite in Wisconsin and the Middle West, many householders are turning to coke for their heater fuel. It is relatively smokeless, and clean, but requires more attention than anthracite. It has about the same ash content as anthracite, the content of moisture and volatile matter is small, and while it is more difficult to ignite, it burns quite freely when once started.

The preceding pages give the classification of the various grades of fuel, but it must be borne in mind that there is a wide variation in the characteristics of the fuel in any one class. For instance, the bituminous coals of Illinois (see Table II), although they may appear almost exactly alike, vary greatly as to their heating qualities, chemical characteristics, convenience in handling, and price to the consumer. This invisible variation in coal puts the domestic consumer at the mercy of his coal dealer, since he is not able, without costly analyses, to determine whether the coal delivered to his bin

was actually mined in the district from which he desired to obtain his supply.

In the larger plants, where the amount of coal used warrants it, analyses of the coal are regularly made, and the coal contracts are based upon the results of the analyses. This insures a just return for the money expended. Purchasing coal on the heat unit basis is equivalent to purchasing so many heat units instead of so many tons of coal.

Value of Proximate Analysis

The true test of any coal lies in its burning, but an analysis gives a reliable indication of what may be expected from the use of a coal, and indicates the type of furnace best adapted for that particular fuel. Furthermore, a knowledge of the chemical contents makes it possible to determine whether or not the coal delivered is as specified. The constituents for which the coal is analyzed are moisture, volatile matter, fixed carbon and ash. The heat value of a coal can be calculated from the coal contents, or, as is usually the case, it can be determined by burning a portion of the sample in a calorimeter so as to determine the heat value directly. The different bituminous coals vary greatly in heat value, some of them ranging as high as 14,700, and others as low as 9,000 British thermal units per pound, a British thermal unit (B. t. u.) being the quantity of heat required to raise the temperature of one pound of water through one degree Fahrenheit.

The analyses shown in Table II give the characteristics of various coals available for use in Wisconsin.

The moisture in coal is undesirable, for it not only reduces the heat value per pound of material fired, but adds to the transportation expense per B. t. u. delivered; it also decreases the furnace and boiler efficiency, since it becomes superheated steam, thereby absorbing heat, which is carried up the chimney with the flue gas. Roughly, the loss of heat value of dry fuel is about one-tenth of one per cent for each per cent of moisture present.

TABLE II

CHARACTERISTIC ANALYSES OF SOFT COALS MINED IN ILLINOIS, INDIANA, OHIO, PENNSYLVANIA, AND WEST VIRGINIA

Location	Volatile matter per cent	Fixed carbon per cent	Moisture per cent	Ash per cent	B. t. u. per lb. as fired
Illinois					
Big Muddy	31.74	48.12	9.10	11.04	[illegible]
Franklin County	[illegible]	[illegible]	9.21	8.71	[illegible]
Murphysboro	[illegible]	51.02	9.28	5.72	[illegible]
LaSalle	[illegible]	[illegible]	16.18	7.08	[illegible]
Wilmington	[illegible]	45.7	10.74	10.20	11417
Indiana					
Clay County	32.66	[illegible]	15.38	5.88	11680
Greene County	33.54	[illegible]	13.53	7.55	11738
Sullivan County	35.17	43.73	12.14	[illegible]	11516
Ohio					
Hocking Valley	30.26	53.19	4.53	12.02	12314
Hocking Valley	36.97	49.91	5.10	8.02	[illegible]
Pittsburg No. 8	33.34	47.52	3.96	15.18	[illegible]
Pennsylvania					
Youghiogheny	36.76	57.45	1.74	4.05	14486
Youghiogheny	31.70	57.00	1.37	9.93	13317
Bessemer	[illegible]	58.96	1.15	8.46	[illegible]
West Virginia					
Pocahontas	22.26	71.75	1.27	4.72	14872
Pocahontas	18.1	74.52	.73	6.65	14589
New River	17.84	75.41	1.74	5.01	14994
Iroquois	27.96	63.45	3.19	5.4	14201

The ash detracts from the value of coal in a number of ways. The greater its percentage, the more difficult it is to obtain complete combustion because of its tendency to pack and obstruct the passage of air through the grate; also the greater may be the proportion of coal lost through the grates with the ash, and the less is the capacity of a given furnace because of the reduction of combustible per square foot of grate area. Useless expense is involved in transporting this inert matter in the coal and in the transportation and disposal of the ash itself.

The ratio of volatile matter to fixed carbon determines the type of furnace best adapted for any particular fuel. Coals in which the volatile matter is proportionately very high usually give very long flames, and cannot be burned completely or smokelessly unless used with furnaces of proper type, size, and proportions, and unless special means are provided for regulating the air supply above the grate.

The Value of Sizing and Washing Coals

The size of coal used in a furnace is an item of considerable importance, though one often overlooked. In general, in using the same coal with a given furnace and draft, the efficiency and capacity of a grate will vary with the size of the coal. Uniformity of size insures a more even distribution of air through the grate, and permits easier control of the fire.

The size best adapted for a given case is dependent on the intensity of the draft, the kind of stoker and grate, and the method of firing. The smaller the coal, the greater, as a rule, is the percentage of ash. This results from the process of mining in which the foreign matter from the roof or floor of the mine naturally finds its way into the smaller coal. Separation into different sizes is accomplished by sending the coal over screens having holes of the proper size.

Bituminous coal is sized into four sizes that differ somewhat in different localities. They average about as follows: lump, all sizes over 3 inches; egg, all sizes between $1\frac{1}{2}$ and 3 inches; nut, sizes between $\frac{3}{8}$ and $1\frac{1}{2}$ inches; and slack, $\frac{3}{8}$ inch and less.

Washing is sometimes employed in preparing the smaller sizes of coal. Coal screenings contain anywhere from 5 per cent to 25 per cent of ash, and 1 per cent to 4 per cent sulphur. Washing eliminates about 50 per cent of the ash and some of the sulphur, which results in a higher heating value per pound. Many coals otherwise worthless as steam coals are rendered marketable in this way.

COMBUSTION

The Principles of Smokeless Combustion

The subject of combustion should be familiar to all mechanical engineers, but unfortunately it is not familiar to a majority of the men who handle the coal in our homes and in our power plants. Therefore, as a basis for a thorough understanding of the problem of smokeless combustion, the theory will be taken up in as elementary a manner as possible.

Every particle of matter is an elementary substance, a compound substance, or a mechanical mixture. An elementary substance is composed of only one element, and is therefore not formed through chemical combination. Silver and gold and the gases known as oxygen, hydrogen, and nitrogen are elementary substances. A compound substance is formed by the combination of two or more elements. Water is a compound, formed by the chemical combination of two elements, hydrogen and oxygen. Any compound can be decomposed into its elements. A current of electricity passing through water will decompose it into its elements, hydrogen and oxygen.

It is impossible to decompose an elementary substance, but its form can be changed by combining it chemically with one or more other elements. The important elements in the study of the combustion of coal are the gaseous elements, oxygen, hydrogen, and nitrogen—and the solid elements, carbon and sulphur.

A mechanical mixture may be composed of two or more elements, of two or more compounds, or of elements and compounds mechanically mixed, but not chemically combined. The air of the atmosphere is a mechanical mixture composed principally of the elements oxygen and nitrogen.

The smallest quantity of an element or a compound that is capable of separate existence is taken as a physical unit of matter, and is called a molecule. Molecules are composed of atoms of elements. An atom is the smallest part of an element that can enter into a compound or be expelled from it.

When two or more elements combine chemically, they form a compound unlike any of the elements, and a definite amount of heat is always produced. "Combustion" may be defined as a rapid chemical combination resulting in heat and light. The substance that is formed by the chemical union is called the product of combustion; and the heat that is produced by the combustion of one pound of the fuel is called the heat of combustion. The substance with which the oxygen combines is called the combustible of the coal; the oxygen supports the combustion. The principal combustibles in coal are the carbon and hydrogen. Combustion is said to be perfect when the combustible combines with all the oxygen that it is capable of combining with; if it combines with less than that amount it is said to be imperfect. Thus, the burning of carbon to form the gas carbon dioxide (CO_2) gives perfect combustion, since CO_2 cannot unite with more oxygen. Burning carbon to carbon monoxide (CO) gives imperfect combustion, since CO can be burned to CO_2 by uniting with another atom of oxygen. As a result of the complete combusion, the heat developed is 14,600 B. t. u., while the imperfect combustion to CO develops only 4,450 B. t. u.

Elements always combine in definite, invariable proportions. For example, two atoms of hydrogen always combine with one atom of oxygen, and the combination forms water. A knowledge of the relative weights of these elementary atoms gives a direct means of computing the amount of oxygen required for combustion. The weights are as follows:

Hydrogen (H)	1
Carbon (C)	12
Oxygen (O)	16
Sulphur (S)	32

In burning carbon (C) to carbon dioxide (CO_2), one atom of carbon of weight 12 combines with two atoms of oxygen, each of weight 16, which combination can be expressed by

$$12 \text{ of } C + (2 \times 16) \text{ of } O = 44 \text{ of } CO_2$$

Dividing this by 12 gives

$$1 \text{ of } C + 2\tfrac{2}{3} \text{ of } O = 3\tfrac{2}{3} \text{ of } CO_2$$

Thus, if 1 lb. of carbon unites with $2\frac{2}{3}$ lb. of oxygen, the result is $3\frac{2}{3}$ lb. of carbon dioxide. In actual furnace practice, the supply of oxygen necessary for combustion is obtained from the atmosphere, which is a mechanical mixture composed of 77 per cent of nitrogen and 23 per cent of oxygen by weight. Since air contains only 23 per cent oxygen by weight, the weight of air required for the complete combustion of 1 lb. of carbon is $100/23 \times 2\frac{2}{3} = 11.52$ lb. Since air is composed of 77 per cent nitrogen, $77/100 \times 11.52$ gives 8.85 lb. of nitrogen which must pass through the furnace for each pound of carbon burned. Table III gives certain data on the reactions and results of combustion for elementary combustibles and the compound carbon monoxide.

At the ordinary temperature and pressure of the atmosphere, a pound of air has a volume of about 13 cubic feet. Using this value, the theoretical volume of air required for the complete combustion of one pound of carbon is

$$11.52 \times 13 = 150 \text{ cubic feet.}$$

The above discussion pictures ideal perfect combustion of carbon. Each atom of carbon will meet with two oxygen atoms at a temperature sufficiently high for ignition. No more air will be supplied than is just sufficient to furnish the exact number of oxygen atoms, and no carbon atoms will pass out of the furnace without finding oxygen atoms with which they can combine. The same is true of hydrogen and sulphur atoms in fuel.

These ideal conditions are not met in actual practice. The varying resistance to the passage of air through the fire, owing to the irregular thickness of the fire on the grate and to lumps of coal, fine coal, clinkers, ashes, etc., tends to prevent intimate contact of the oxygen and the fuel. Under these conditions, if just the theoretical amount of air were to pass through the fire, it is evident that enough of its oxygen would not combine with the fuel to burn the fuel completely. Some of the carbon would burn to carbon monoxide instead of carbon dioxide, and the fuel would not develop its full heat value. This condition is always indicated by a lowering of the furnace temperature, as well as the production of

TABLE III

OXYGEN AND AIR REQUIRED FOR COMBUSTION

By Weight

1	2	3	4	5	6	7	8	9	10
Oxidizable Substance or Combustible	Chemical Symbol	Atomic or Combining Weight	Chemical Reaction	Product of Combustion	Pounds of oxygen per pound of column 1	Pounds of nitrogen per pound of column 1 (3.32*×O)	Pounds of air per pound of column 1 (4.32†×O)	Pounds of gaseous product per pound of column 1 (1+column 8)	Heat value per pound of column 1 (B. t. u.)
Carbon..........	C	12	$C+2O=CO_2$	Carbon dioxide	2.667	8.85	11.52	12.52	14600
Carbon..........	C	12	$C+O=CO$	Carbon monoxide	1.333	4.43	5.76	6.76	4450
Carbon Monoxide	CO	28	$CO+O=CO_2$	Carbon dioxide	.571	1.90	2.47	3.47	10150‡
Hydrogen	H	1	$2H+O=H_2O$	Water	8.0	26.56	34.56	35.56	62000
Sulphur.........	S	32	$S+2O=SO_2$	Sulphur dioxide	1.0	3.32	4.32	5.32	4050

* Ratio by weight of O to N in air.
† 4.32 pounds of air contain one pound of O.
‡ Per pound of C in the CO.

smoke with bituminous coal and by the amount of carbon monoxide discharged from the stack with anthracite or coke. Therefore, the theoretical quantity of air must be increased by an amount that will be sufficient to furnish enough oxygen for complete combustion under furnace conditions. This excess is usually 50 per cent, and may reach 100 per cent. The amount varies with the draft, the kind of coal, and the method of firing the coal. For example, while only 11.52 lb. of air are theoretically required for the complete combustion of 1 lb. of carbon, it is usually necessary to furnish 18 to 24 lb. On the other hand, a supply of air that is more than sufficient is a source of waste, as this extra air merely absorbs heat and carries a good portion of it up the stack. This extra excess of air may also have the effect of cooling the furnace below the ignition point of the fuel. Carbon and oxygen atoms will not unite unless the ignition temperature is reached, and it is therefore necessary to keep the furnace at a high temperature at all times.

The following experiments[2] will serve to illustrate the principles of combustion which have been set forth above:

EXPERIMENTS IN SMOKE PRODUCTION

"The laws of combustion are the same whether applied to a furnace, a lamp, a candle, a gas jet, or a gas stove flame; hence, laws that apply to any one of these, apply to all.

"Pour kerosene oil into a plate and set fire to it; dense, black clouds of smoke will rise, due to lack of sufficient air properly mixed with the gases to burn the oil completely over the whole surface.

"A candle having a small wick produces a clear, bright light without smoke. A candle having a large wick has a dark, yellow-colored flame that has a tendency to smoke and does not give as bright a light as the candle with the small wick. The big wick tends to supply more oil than the conditions of the air supply warrant.

"A torch has a still larger wick, since it is intended to burn a larger supply of oil. It smokes badly because the wick supplies more oil than can be burned smokelessly under the conditions of air supply.

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[2] Cosgrove, J. F., *Coal, Its Economical and Smokeless Combustion*. The Technical Book Publishing Company, Philadelphia, Pa.

"These examples show that there is a limit to the quantity of oil that can be burned smokelessly as a naked light. Consequently, to obtain more light than the candle would give, the kerosene lamp was invented. One reason for the torch smoking is that the wick is so thick that the air supply cannot intimately mix with the gas from the oil brought up through the center of the wick. To obviate this in the lamp, a wide, flat wick is used. Also a chimney is employed to increase the air supply and to direct a current of air upwards and against the flame, thus insuring a sufficient air supply to burn the increased amount of oil and give a clear, white light. If the chimney is removed, the flame will smoke, since there is nothing to create a current of air and direct it against the flame; hence, the air supply is deficient. The student's lamp with its circular burner permits the largest size of wick to be used, and since the air through the center of the tube is heated before coming into contact with the gases, the burner is well suited to give perfect combustion where a large amount of oil is being burned.

"This discussion shows that, in order to have smokeless combustion, the air supply must be sufficient, and must be intimately mixed with the gases of the fuel while at the proper temperature; also, that the furnace must be suitably constructed to burn the required supply of fuel. A candle will give perfect satisfaction for the consumption of a small amount of oil, but an Argand burner is necessary to burn a large supply of oil satisfactorily.

"Every lamp has a range through which it will burn smokelessly, but, without exception, if the range be exceeded, and the wick turned up so far as to supply more oil than can be burned perfectly, the lamp will smoke. This shows that if an attempt is made to burn more fuel in a furnace than the furnace is designed to burn, smoke will result.

"Turn up the flame of a lamp until it produces a clear, bright light, indicating perfect combustion. Gradually close the draft openings at the base of the burner, and watch the flame. As the air supply is diminished, the flame gradually lengthens and becomes darker and darker, until, finally, it begins to smoke; then, if the air is still further restricted, the smoke increases, and when the flame extends above the chimney a stream of dense, black smoke arises from the flame. The effect will be the same whether the air supply is restricted at the burner or at the top of the chimney, the amount of smoke produced being in proportion to the restriction of the air supply.

"This experiment shows the effect of a restricted air supply on the flame and on the smoke produced. The effect will be the same whether the air supply is cut down below the requirements of the fuel, or the fuel supply is increased above the capacity of the air supply.

"The flame of a gas stove when properly adjusted is short and produces a feeble, bluish light, similar to that of a Bunsen burner. Gradually restrict the air supply, and the flame will gradually lengthen and give off less and less heat; when the air supply is restricted too far, the flame will assume a dark-yellow color and will smoke, owing to the particles of free carbon that the restricted air supply allows to escape unburned. Note the time necessary to boil a quart of water from the same temperature, first, when the flame of the gas stove is short, due to a proper air supply, and second, when the flame is long and yellow, due to a restricted air supply; the short flame will be found to give a great deal more heat than the long flame, since it utilizes all the heat of the gases.

"Turn the wick of a lamp up until the flame burns brightest, and remove the chimney. The flame will smoke badly. The oil from the wick is converted into a gas by the heat of the flame, and if the gas were mixed with air in the proper proportions, a very hot, non-luminous flame would result, similar to that of a Bunsen burner. The luminosity of the lamp flame is due to the fact that the air does not penetrate within, and mix with, the gas supply. The air simply envelops the flame so that all the combustion is on the surface of the gas supply. As the oil is vaporized, it becomes heated and the hydrocarbons are dissociated. The free carbon floating in the gas supply is then heated to a white heat and travels to the surface of the gas, where it combines with oxygen and burns smokelessly. The chimney produces a current of air and deflects it against the flame, thus intimately mixing the air and the gas at the source of the flame. When the chimney is removed, the means of producing sufficient air for the gas and of intimately mixing the air and the gas is absent; hence, a smoky flame results.

"A gas-light burner tip is designed so to spread the flame as to give complete combustion with full luminosity. Remove the tip so that the gas issues from a round orifice, and a long, very smoky flame will result, because the air is not intimately mixed with the gas, and more time and space are needed for the burning of the gas.

"These examples indicate the importance of an intimate mixture of sufficient air with the gaseous content of coal, in order to obtain complete combustion and to generate the maximum quantity of heat from the gas without smoke.

"Lower a pan of cold water into the flame of a candle that is burning brightly without smoking, and the flame will smoke badly, owing to particles of free carbon in the flame coming in contact with the cold metal and being chilled before they can burn. This illustrates why a furnace smokes badly when the heating surface is arranged so that the flames can come in contact with it.

"A pan of cold water placed on a gas-stove flame does not cause smoke, because the burner mixes the air and the gas before they

reach the flame; hence, the carbon of the gas is completely burned as it is dissociated and there are no free particles of carbon in the gas to produce smoke when the pan cools the flame.

"Once a flame starts to burn, the heat produced by the combustion of the gas is sufficient to maintain the phenomenon of burning. If the flame is suddenly cooled at any point, the combustion beyond that point is at once arrested. A fine wire gauze held midway in a gas flame will cool the flame, since the incandescent gas cannot pass through the meshes of the gauze without being cooled below the igniting temperature by contact with the metal, which is a good conductor of heat. Hence, no combustion takes place above the gauze, although smoke is produced and the unburned gas passes through freely. If the gas above the gauze is ignited, it will burn. Also, if the gauze is held in a jet of gas that is escaping unlighted, the gas above the gauze may be burned without the gas below the gauze igniting.

"These experiments show that if the combustion chamber of a furnace is not of sufficient capacity to prevent the flame from coming in contact with the cool surfaces of the boiler, dense, black smoke is sure to result."

Having outlined the essential features of perfect combustion, let us turn our attention to the manner in which these conditions can be obtained in practice. These conditions are clearly and forcibly set forth by Hirshfeld and Barnard[3] as follows:

CONDITIONS FOR COMPLETE AND SMOKELESS COMBUSTION

"(a) If air is passed upward through a deep bed of ignited carbon devoid of volatile matter, there is a tendency for any CO_2 that is formed in lower layers to be reduced to CO when coming in contact with the carbon above. If this CO is not subsequently supplied with a proper *amount of air* while still at a high temperature it will pass off unoxidized, and this will result in a loss of heat which would otherwise be made available. It is, therefore, important that an adequate air supply and a suitable *temperature* be maintained in the upper part of, and just above, the bed of fuel. This air may either pass through the bed or be supplied from above.

"The foregoing applies, of course, to the combustion of coke and charcoal as well as to carbon. Anthracite coal, which is mostly fixed carbon, behaves similarly, but in this case there is also a small amount of volatile matter which must be properly burned. These fuels, which have little or no volatile matter, give *short flames* above

[3] Hirshfeld and Barnard, *Elements of Heat Power Engineering*, John Wiley & Sons, New York.

the fuel bed, the flames being due to the combustion of CO and the small quantity of volatile matter present.

"(b) When coal possessing a considerable amount of *volatile matter* is placed on a hot bed of fuel, the greater part of the volatile portion distills off as the temperature rises, and the residue, which is coke. burns in the manner just described. The more serious problem that confronts the engineer in this case is the complete oxidation of the combustible part of this volatile matter. Evidently in the ordinary up-draft furnaces that are fired from above the combustion of this part of the fuel must occur above the fuel bed, just as is the case with CO; and in order that the combustible gases may be completely burned, the following four conditions must exist:

"(1) There must be *sufficient air* just above the fuel bed, supplied either from above or through the fuel bed itself; (2) this air must be properly distributed and intimately *mixed* with the combustible gases; (3) the mixture must have a *temperature* sufficiently high to cause ignition (some of the combustible gases, when mixed with the burned gases present above the fuel, have an ignition temperature of approximately 1450° F.); and (4) there must be sufficient *time* for the completion of combustion, that is, the combustion must be complete before the gases become cooled by contact with the relatively cold walls of the boiler (which are at a temperature of about 350 degrees) or with other cooling surface.

"(c) To prevent the *stratification* of the air and gases, special means are sometimes adopted, such as employing steam jets above the fire and using baffle walls, arches, and piers in the passage of the flame, to bring about an intimate mixture.

"(d) In order that the air used above the fuel bed shall not chill and extinguish the flame, it should be *heated* either by passing it through the fuel bed, or through passages in the hotter parts of the furnace setting, or in some other way before mingling with the gases; or else the mixture of gases and air should be made to pass over or through hot portions of the fuel bed, or should be brought into contact with furnace walls, or other brickwork, which is at a temperature sufficiently high to support the combustion.

"(e) In order that the flame shall not be chilled and extinguished by coming in contact with cold objects, it should be protected by the hot furnace walls until combustion is complete. The *furnace* should have proper volume to accommodate the burning gases, and, when the conditions are such that the flame is long, the distance from the fuel bed to the relatively cold boiler surfaces with which the gases first come in contact, should be at least as great as the length that the flame attains when the fire is being forced. The *length of flame* depends on the amount and character of the volatile matter in the fuel, on the rapidity of combustion, and on strength of draft. It varies from a few

inches, with coke and anthracite coal, to 8 feet or even more with highly volatile coals—even 20 feet has been reached with some western coals.

"(f) In order to have complete combustion of *all* the fuel in a furnace it is necessary that uniform conditions prevail throughout the fuel bed; and to bring this about it is essential that the fuel itself be uniform in character. Therefore, the best results are obtained with coal that has been graded as to *size*. Especially is this true with anthracite coal which ignites slowly and is more difficult to keep burning than volatile coals. This coal requires a rather strong draft and unless the bed is uniform the rush of air through the less dense portions tends to deaden the fire in those regions, hence good results can be obtained with this coal only when it is uniform in size and evenly distributed."

It is thus seen that the whole problem of burning soft coal resolves itself into one of burning the volatile matter satisfactorily. The conditions to be met can be more briefly summed up as follows:

In order that combustion may be smokeless and efficient, the volatile gases and separated free carbon must be brought into contact with the proper quantity of air and maintained at a temperature above the ignition point until oxidation is complete before they are brought into contact with the heat absorbing surfaces of the boiler. Mere excess of air will not effect smokeless combustion, even if the gases are thoroughly mixed, if the temperature is prematurely reduced below that necessary for combustion by contact with the heat absorbing surfaces of the boiler.

FIRING FOR POWER PLANTS

Methods of Hand Firing Soft Coal[4]

The hand firing in many of our smaller plants violates all the principles laid down for insuring good combustion. The construction of many of these furnaces is such that it is almost impossible to operate the plant without smoke, but often the result depends more on the fireman than on the design of the furnace. The chief difficulty with hand fired furnaces lies in the intermittent nature of the firing. Very few firemen can be induced to fire regularly and frequently, because it is easier to put in enough coal to last 20 or 30 minutes at one time and have little or nothing to do in the interval between firings. When coal is supplied in such large quantities at long intervals, the result is that at the time of firing the temperature of the furnace is lowered, the resistance to the flow of air through the fuel bed is increased, and consequently a great quantity of volatile matter is generated which cannot be burned for the lack of air and the necessary amount of heat.

A study of the requirements for complete combustion and a desire on the part of the fireman to obtain good results will do more to conserve our fuel supply and to clear the air in cities than any other one influence.

Four methods of hand firing coal are more or less in general use: (a) the spreading, (b) the coking, (c) the alternate, and (d) the spot.

Spreading Method—The spreading method is most commonly adopted for anthracite coal. It consists in spreading the fresh coal evenly in a thin layer over the whole grate. With soft coal this is the least efficient method, and it produces the most smoke. Soft coal requires more air immediately after

[4] For a more thorough discussion on hand firing soft coal, see *Hand Firing of Soft Coal Under Power Plant Boilers*, by Kreisinger, Henry, (Technical Paper No. 80, Bureau of Mines), an excellent paper that should be in the hands of every fireman.

firing than does anthracite coal, and covering the entire fire not only decreases the flow of air through the fuel bed, but lowers the temperature of the furnace enough so that a large part of the volatile matter will pass off unburned, resulting in a large heat loss and much smoke.

Coking Method—The coking method of firing is best adapted for caking coals and for where the demand upon the furnace is fairly regular. It is not a flexible method, and therefore is not suitable for conditions under which the load varies quickly. It is quite effective when used in firing under boilers used for heating purposes.

In this method the coal is first piled on the dead-plate near the fire door and allowed to slowly coke. This insures a slow and uniform distillation of the volatile matter. The volatile matter, passing back over the glowing coals on the grate, mixes with the hot air passing through this portion and is completely burned. When the coal is first placed upon the dead-plate, the amount of volatile matter given off may be greater in amount than can be combined with the air passing through the fuel bed. In order to supply the proper amount of air at times of firing, there should be means, through holes in the furnace door, of supplying air over the fuel bed. This air, entering in small streams through the fire door, mixes with the gases and supplies the extra oxygen needed. When the coal has been thoroughly coked, it is pushed back onto the fuel bed, and spread evenly over the surface, care being taken to fill any thin spots or holes in the fire. A new charge is then put on the dead-plate. Large lumps should be broken up before being placed on the dead-plate, and the crust formed during coking should be broken up as frequently as necessary.

Alternate Method—The alternate method is best adapted for non-caking coals, such as the Illinois coals, and when properly used will give excellent results, even with coals rich in volatile matter. For small furnaces, this method consists of firing alternately, lengthwise, on one-half of the fuel bed at a time, at such intervals as may be necessary to hold the steam pressure. Depending on the rate of driving, these intervals will vary from 3 to 8 minutes. This method always leaves half the

fire bright to furnish heat required to burn the gases coming from the coal which has just been fired on the other half. Also this method allows the air to pass freely through the bright half of the fire, thus becoming heated and suitable for mixing with the volatile matter which is passing off from the coal which has just been fired.

Spot Method—Alternate firing is called spot or checker firing when applied to a fuel bed area which is large. Spots on an imaginary checkerboard are fired alternately, and, as before, the volatile matter from the fresh coal is supplied with heated air by the excess amount that passes through the remaining portions of the bed. In this method of firing, the coal is placed each time on the brighter and thinner portions of the fuel bed. Thin spots will occur even with the most careful firing, because the coal never burns at a uniform rate over the entire grate area. Where the air flows freely through the fuel bed, the coal burns faster than in places where the flow of air is less. If the firings are too far apart, the coal in the thin spots may burn out entirely and a large excess of air enter the furnace in streams. This air usually passes through the furnace without mixing with the gases from the coal, and deprives the boiler of considerable heat.

Reasons for Small and Frequent Firings

Evidently the best results can be obtained only when the amount of air supplied varies directly with the weight of coal fired. In intermittent firing, such as hand firing, if a large amount of fresh coal is thrown on the fuel bed, it chokes the air supply at a time when it is most needed to mix with the volatile matter which is distilled from the coal. Therefore, immediately after firing a large quantity of air should be admitted over the fire and then gradually reduced as the distillation of the volatile matter nears completion. The air supplied over the fire should be admitted in small streams, so that the air will be heated as quickly as possible and will be more thoroughly mixed with the gases. The distillation process is usually completed in two to five minutes after firing, and only a comparatively small amount of air need be

admitted over the fuel bed after the process is finished. This means a very wide variation in the air supply in a short period of time. It is almost impossible to obtain such regulation in practice, and the only way out of the difficulty is to fire coal in small quantities at short intervals of time. In this way the distillation of volatile matter becomes more nearly constant and at all times more nearly proportional to the air supply. The ideal case is illustrated in the use of mechanical stokers by which the coal is fed into the furnace continuously at a uniform rate.

Further Aids and Precautions in Firing

1. A suitable damper should be placed in the uptake or in the breeching leading from each boiler and arranged so that it can be operated by a system of levers from a convenient point near the furnace door. A damper without a proper connection for manipulation by the fireman is as bad as an engine throttle without a hand wheel. Damper connections for hand manipulation cost comparatively little and are usually easy to install.

2. Each boiler should be fitted with draft gages. These gages should be placed at the front of the boiler near the damper connection, so that the fireman can read the draft while adjusting the damper. One, connected to the furnace above the fire, will indicate the "drop" of draft[5] through the fire, while the second, connected to the breeching below the damper, will indicate the "drop" through the tubes, etc. The fireman should become familiar with the amount of draft required with a clean fire and clean tubes so that he can readily note any change that takes place in the fire or tube conditions by the change in the draft readings. An increase in the drop of draft through the fire indicates that the fire is becoming dirty or that it is too thick; a decrease indicates holes in the fire or a fire that is too thin. An increase on the gage connected to the breeching indicates increased friction due to deposits of soot, ash, etc., upon the tubes and baffles, and a de-

[5] Draft is defined as the difference in pressure which produces the motion of the gases.

crease indicates reduced friction, probably due to a broken or burned out baffle wall. The importance of proper draft regulation cannot be too strongly emphasized. It has been said that the average housewife pays more attention to draft regulation than is customary in many of our small plants. The greatest gain in economy in the boiler room is obtained by proper draft regulation. Perfect hand control is physically possible, but it is never obtained in practice, because the fireman is not on the job every minute to see that such proper regulation is obtained. Therefore, to get the highest economy out of the plant, some means of automatic draft control should be installed. The saving in fuel will pay for such an installation in a very short time.

3. Regulate the draft by using the damper in the uptake or breeching. Regulation of draft with ash pit doors is objectionable and should be avoided. The position of the damper should be changed gradually and just enough to change the air supply to the desired rate of combustion. To close it the full amount quickly will produce dense smoke.

4. The ash pit door should be kept open and the ash pit bright at all times. If the pit becomes dark, it is evident that the fire is getting dirty and needs cleaning, which, if not done, will cause imperfect combustion and smoke.

5. The best results are obtained if the fires are kept level and rather thin. The best thickness is 4 to 10 inches, depending on the character of the coal and the strength of draft.

6. Do not allow ashes to collect in the ash pit. They not only shut off or create an uneven air supply, but may cause the grate to be burned.

7. Avoid excessive disturbance of the fuel bed, as this may cause troublesome clinkers. Every time the fire door is opened the excessive air which enters the furnace reduces the furnace temperature and causes loss of heat.

8. With frequent firings there is much less danger of holes forming in the fuel bed. The thin spots are seen and covered with fresh coal before the holes actually form, which reduces the loss from excess air.

9. In cleaning the fire the clinker and ash should be removed in such a way as to waste as little of the combustible as pos-

sible. This is best accomplished by what is known as the side method of cleaning, in which one side of the fire is cleaned at a time. The good coal is scraped and pushed from the side to be first cleaned to the other side; the separation is easily made because the clinker and ash naturally sink to the grate, while the good coal remains on top. The clinker and ash are then removed through the fire door. When one side of the furnace is cleaned, the burning coal from the other side is scraped over to the clean side and the clinkers and ash removed as before. The condition of the fire at the beginning of cleaning should be such that there will be sufficient fire in the furnace to start a hot fire quickly when the cleaning is completed. During cleaning, the damper should be partly closed to avoid the rush of too much cold air through the furnace and tubes.

10. The settings of all boilers should be air-tight in order that no air can enter the boiler or furnace except under the grate or other places under control of the fireman.

11. The Bureau of Mines recommends that all boilers should have the soot blown off the tubes every day when in operation.[6] In addition, fire tube boilers should have the tubes scraped twice a month, particularly if the fuel is sooty. A boiler should be blown off at least once a day, preferably in the morning before starting the day's run. The mud has then settled and can be removed more easily. A boiler should be washed thoroughly on the inside every two to four weeks, the time between washings depending on the quality of the feed water.

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[6] Kreisinger, Henry *The Hand Firing of Soft Coal under Power Plant Boilers*, (Bureau of Mines, Technical Paper No. 80).

FIRING FOR DOMESTIC HEATING

Bituminous Coal in House Heating Furnaces

Stoves, ranges, house heating boilers, and hot-air furnaces are, as a rule, intended for the use of anthracite coal or coke. Whenever bituminous coal is burned in furnaces designed for anthracite, all the principles of combustion are violated, and smoke results, especially if the same method of operation is used as for anthracite. When anthracite coal or coke is burned, a relatively small space above the fuel bed is required because combustion takes place in or close to the fuel bed. When bituminous coal is burned in this same furnace, the volatile matter that leaves the surface of the coal is rapidly cooled in passing over the heat absorbing surfaces between the fire pot and the smoke pipe, and the temperature of the volatile matter is quickly brought below that necessary for its ignition. This results in the flue surfaces becoming coated with soot and in large heat losses.

Cleanliness and convenience of operation have made anthracite the most desirable house-heating fuel, but its gradual increase in price has caused many people who burned nothing else to feel that the convenience resulting from its use costs too much, and they have changed to bituminous coal. Until the last five years, makers of house heating boilers and furnaces gave little thought to the fuel to be burned, but they have come to realize the importance of burning bituminous coal, and many desirable house heaters for this purpose are appearing on the market. The householder can realize a considerable saving by burning bituminous coal if he is equipped to burn it properly, and the prospective builder will do well to study the situation carefully in planning the heating plant for his house.

A thorough study of the principles of combustion given in a previous chapter will enable the householder to decide upon the limitations or possibilities of his heater. If he decides to burn bituminous coal, he should realize that to promote economy he must sacrifice convenience. In firing bituminous coal,

the fresh fuel should not be spread over the entire surface of the fire as is usually done with anthracite. Bituminous coal requires more air immediately after firing than does anthracite, and covering the entire fuel bed not only decreases the flow of air through the fuel bed, but lowers the temperature in the fire pot enough to cause incomplete combustion. Comparatively small charges of coal should be put on at frequent and fairly regular intervals.

Use some coking method of firing: that is, work the partly burned coal, from which the volatile matter has been driven, to one part of the fuel bed, and throw fresh coal on the remaining portion. The fresh fuel then ignites slowly, and as the volatile matter is gradually driven off, it is ignited by the brightly glowing portion of the fuel bed. As soon as the fresh coal has been coked it should be spread over the entire fuel bed, where it will burn as a bright fire without further smoke. The smaller the charges fired, the more efficient and smokeless will be the combustion.

Other Precautions for Saving Fuel in Heating a House

Tests have shown that from 40 to 50 per cent, on the average, of the heating value of the coal is usefully employed in heating a house or a building. Under conditions of proper installation and frequent and careful attention, 50 to 60 per cent of the heating value may be converted into useful heat, while under unfavorable conditions, only 25 per cent, or even less, of the heat value of the coal is utilized. The loss may be due to either poor operation or poor installation. Below is given an excellent summary of requirements, taken from a report of the Engineering Experiment Station of the University of Illinois.[7] To this list should be added the advisability of recirculating the air if a hot air system of heating is used, and the use of storm windows:

"The following is a summary of installation and operating requirements to which your plant and your methods of attendance should conform. This list is general, but in so far as it applies to

[7] *The Economical Purchase and Use of Coal for Heating Homes, with Special Reference to Conditions in Illinois*, (Circular No. 4, Engineering Experiment Station, University of Illinois), Urbana, Illinois.

your installation and your conditions of operation, which should be compared with or checked against it, item by item, the answer to each requirement should be either, 'My plant meets that condition, or 'It is operated as here indicated.'

"(1) The chimney should be absolutely tight, and should have a continuous fire clay flue lining from top to bottom. Round or square linings are to be preferred, and there should be no offsets.

"(2) The smoke pipe should grade up to chimney, and should always be straight and short.

"(3) A shut-off or cross damper is required in the smoke pipe to control the intensity of the chimney draft, and there should also be a check draft damper in the smoke pipe beyond the cross damper. Use the check damper for the ordinary daily regulation of the fire.

"(4) There should be a lift damper or slide in fire door, but never use this damper to check the fire. All dampers must fit true and be tight.

"(5) With the check draft closed, the fire should burn up quickly when the draft damper in the ash pit is open; otherwise the draft is deficient.

"(6) The by-pass damper, if provided in your heater, must be tightly closed except when starting fire.

"(7) The heater base must be tight, and grouted to the floor, so that no air leakage into the ash pit can occur at this point.

"(8) Heating surfaces must be kept clean and free from soot and ash accumulations, and the entire ash pit must be cleaned daily.

"(9) Grates must be true and not warped, must move easily, and have no broken places for coal to drop through. Unburned or partly burned coal should not appear in the ashes at any time.

"(10) All basement piping, heater surfaces, and smoke pipe must be completely covered.

"(11) Soft coal should be fired frequently in rather small charges by the alternate or coking method, and all overdrafts should be closed as soon as coking is complete. The fire should not be poked nor stirred from above.

"(12) House heaters must have provision for overdraft through fire door or around top of fire pot.

"(13) The fuel pot must be kept full with fire surface at level of fire door; let ashes accumulate *on* (not under) the grate in mild weather. Grates must not be shaken too long and violently, and clinkers must be removed with as little disturbance of the fire as possible. *Never shake or disturb a very low fire until you have added and ignited a little fresh fuel.*

"(14) Anticipate the heating demand by firing promptly when the outside temperature *begins* to drop, or the wind increases. Never allow a fire to burn too thin or to develop holes in the fire bed.

"(15) If the heater is small for its work, do not use coal containing a large amount of fine material. If the heater is amply large and careful attention can be given to handling the fire, fine material may be used without disadvantage.

"(16) Some kind of automatic damper regulator is essential to economy of operation.

"(17) The house must be kept at a uniform temperature and not allowed to cool down more than ten degrees at night.

"(18) The temperature of all rooms must be as low as is consistent with comfort. To heat a house to 75 degrees F. instead of to 70 degrees F. with an average outside temperature of 40 degrees F. for the entire heating season means a 17 per cent increase in fuel consumption.

"(19) All windows and doors must be as tight as possible.

"(20) The circulation of the air, steam, or water must be uniform and positive to all parts of the system. If unsatisfactory, an experienced steam fitter or furnace man should be consulted.

"(21) The heater as well as the system as a whole must be kept in first class condition, and defects of any sort repaired immediately. Satisfactory operation and sanitation require that all ducts, registers, and radiators be kept scrupulously clean and free from dust, cobwebs, and other accumulations."

DEVICES FOR BURNING SOFT COAL WITHOUT SMOKE

At low rates of combustion, it is possible to obtain with hand firing as complete and smokeless combustion as with any of the devices that are on the market for this purpose, but this involves great skill in handling fire, frequent attention, and considerable experience and judgment on the part of the operator. The ordinary house heating equipments and small power plants do not have this service, but it is possible to use any one of several devices to assist in obtaining the desired results. These may consist of a particular arrangement of setting design in order to approximate conditions specified in a previous chapter, viz., to secure such temperature conditions as will insure complete combustion. Baffle plates or walls may be introduced to intimately mix the combustible gases and the air necessary for their combustion. Some designs introduce coal under the fire or at one side, either manually operated as often as necessary, or automatically and continuously, taking power from a motor whose speed may be regulated by hand or controlled by the changes in the resulting temperature, or by the steam pressure.

No one type of setting or of stoker will handle all kinds of coal with equal success, and for each coal and set of conditions, there is some particular type that will give the best service. In any one of them, the fundamental principles of combustion must be respected before it can be expected to accomplish its purpose. With such a device it is possible, with a little care and experience, to approximate the best results obtainable with the given coal and imposed conditions.

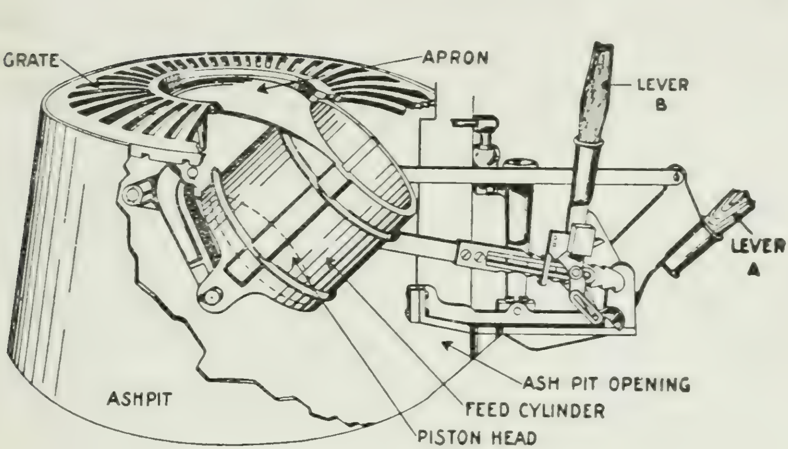
HOUSE HEATING FURNACES

The ordinary domestic furnace or stove is usually a self-contained unit, too small to permit of the introduction of auxiliary devices for smoke prevention, and the steps toward this end must be in the use of proper fuel and in the details of hand firing and control, as outlined in the previous chapter. The ca-

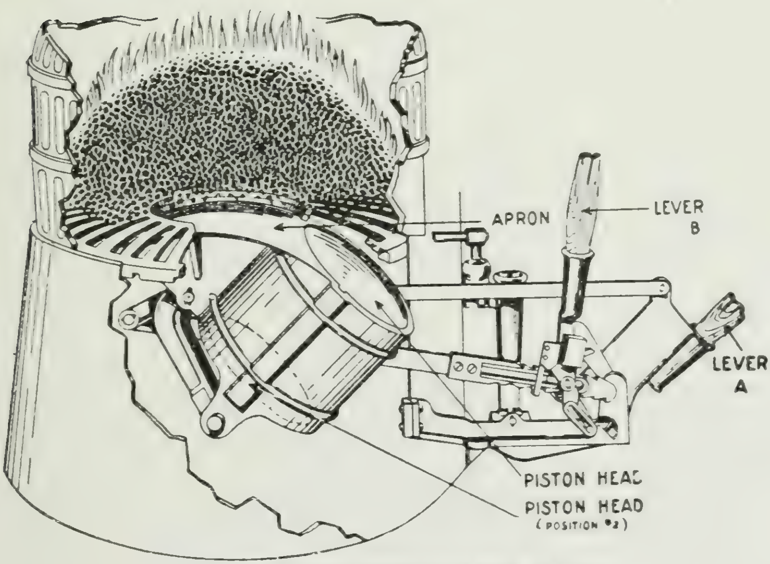
pacities of most of the smaller furnaces are based on the use of good grades of anthracite coal, and if soft coals are used, due consideration must be given to differences in heating value of the coal, the grate surface, air passages, and combustion chamber required, and the more careful and frequent attention necessary.

House heating furnaces for burning soft coal may be divided into three general classes, namely, the underfeed, the magazine feed, and the down draft. For any except the smallest size of furnaces, the feeding of fuel from underneath the fire is one of the most practicable and satisfactory ways of burning soft coal without smoke. The hot fire is always on top, and the radiation, which is the most effective method of heat transfer from a hot body, affects the heat absorbing surfaces directly, and is not choked or smothered by fresh fuel being dumped on top of the fire, as is the case with overfeed furnaces. Then, too, the volatile gases, which contain the most valuable heat producing elements in the combustible, must pass through the hot bed of coals and the temperature is raised to such a point that they will combine with the oxygen of the air supply, thus actually burning what would ordinarily go up the chimney as smoke. To the uninitiated, it would seem impossible for the fire to keep itself going, but one has only to realize that radiant heat is equally effective in all directions, against the draft as well as with it; that it will heat up the green fuel directly below the glowing coals and drive off the volatile gases as readily as though the green fuel were above it. (This is illustrated by a lighted cigar. The air is drawn through the glowing portion, and the tobacco, i. e., the fuel, is volatilized to give the desired smoke, the heat being entirely absorbed by the fuel and by the evaporated moisture in the cigar. A dry cigar burns quickly and gives a "hot" smoke. The tobacco would be burned just as rapidly by blowing out through the cigar, but the volatile matter, i. e., the smoke, would be burned in passing through the glowing tip. It is obvious, then, that the ordinary overfired furnace does the same thing as a properly utilized cigar,—makes smoke,—though the purpose of the operation is entirely different.)

The direct saving, then, is in burning what would ordinarily pass off as unburned or only partially burned combustible. Coals with a high ash content or that have such constituents



READY FOR COALING



CYLINDER TILTED FORWARD AFTER COALING

Fig. 2
Underfeed Type of House Heating Furnace.

in the ash as to fuse and clinker so that they cannot be easily used in the overfeed furnace, can often be used to advantage in the underfeed type, because the clinker forming materials are usually oxidized to a quite fine ash or powder that can be shaken through the grate.

The underfeed devices are usually of two types: one has a direct-acting plunger or ram, which forms the bottom of a pot or hopper. This hopper is below the grate, can be filled conveniently, and by proper mechanism brought to a position under the fire. The ram is then raised by direct leverage or by a ratchet device until it is even with the top, when the hopper is tipped forward, the coal being held up in the fire chamber by an apron on the back side of the hopper. The ram is let down again, and the process repeated as often as the judgment of the operator dictates. Fig. 2 is an illustration of a typical hot air furnace of this form. The other type uses a screw conveyor, taking fuel from the bottom of a previously filled hopper or magazine, and forcing it up under the fire.

Fig. 3 is an illustration of a typical house and apartment heating boiler and furnace of the magazine feed type. The magazine is usually designed to hold a sufficient supply of coal to run the boiler from twelve to eighteen hours without attention. The magazine is charged through the fuel door at the top, the fuel gradually working downward at a rate proportional to the demand upon the boiler. The gases generated from the fresh coal are drawn through the fire and burned. A small amount of air is allowed to enter through the grates, but the main air supply is through the draft door, this air passing through the unburned fuel first after being preheated in passing over the baffle plate at the draft door. Usually passages are provided leading from the top of the magazine to the hot part of the fire below. Any gases forming on top of the coal are thus brought down to the fire and burned.

Larger sizes of steam or hot-water heating plants, such as are used in large residences, apartment houses, churches, and store buildings, may use another device which is quite as effective as the above named types, known as the down-draft furnace. This device has a double grate, one above the other, the space be-

tween forming a combustion chamber. The fuel is shoveled onto the upper grate, which is covered with a layer of live coals from kindling or previous fire. The major part of the air necessary for combustion is allowed to enter through the doors to the upper grate. The distillation of the volatile matter of the coal occurs in the same manner as in the underfeed furnace described above, with this difference, however,—the spaces of

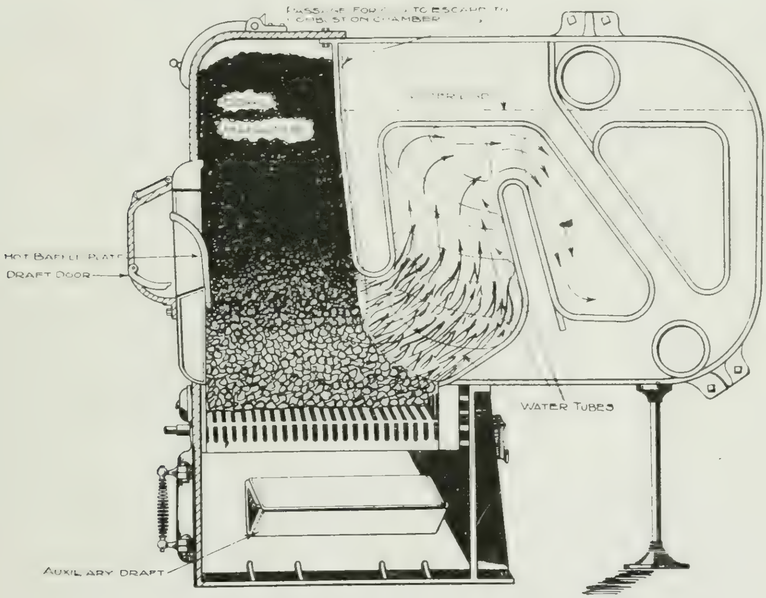


FIG. 3
MAGAZINE TYPE OF HOUSE HEATING FURNACE.

the upper grate are purposely designed quite large, so that as the coal becomes coked and is broken up by its own heat or reduced in size by burning away of its carbon, some of the glowing coke falls through onto the lower grate, through which a very limited amount of air is allowed to come from below. The mixture of highly heated volatile gases and air coming through the upper grate strikes the glowing bed of coke on the lower grate, which, in turn, has been kept hot by its own supply of air, and the conditions for complete and smokeless combustion,

as described in a previous chapter, are fulfilled. All of the combustible part of the coal is thus completely burned in this combustion space, the resulting hot gases going to the heat absorbing surfaces or elements of the steam or hot water system.

Fig. 4 shows a down draft furnace of the larger type. Down draft cast iron and steel heating boilers for large residences and apartments are now being manufactured by a number of different makers.

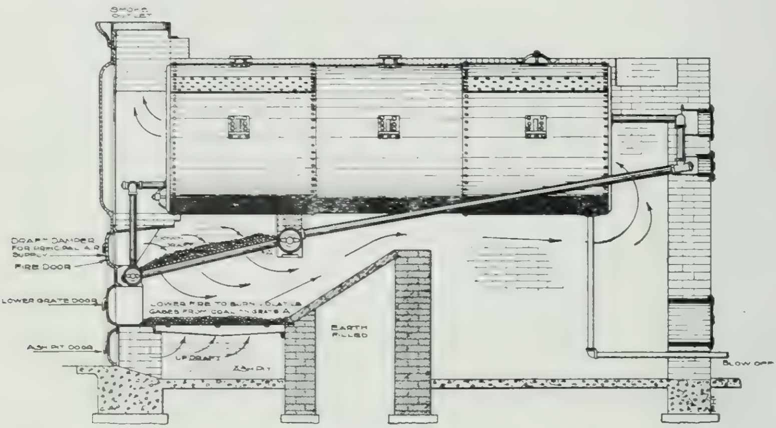


FIG. 4

FIRE TUBE BOILER EQUIPPED WITH DOWN DRAFT FURNACE.

The requirements to be met by furnaces in order to give complete and smokeless combustion have been set forth in another chapter of this bulletin. With a knowledge of these requirements, any prospective builder should be able to select his furnace intelligently.

POWER PLANT FURNACES

In the consideration of the larger heating and power plant boilers, we find a large number of devices for use in burning soft coal without smoke, or at least to keep the smoke to such a small amount that it is not objectionable. An investigation by the United States Government, (Bulletin No. 40, Bureau of Mines, "*Smokeless Combustion of Coal*", p. 100), showed that

"the total number of steam plants having boilers fired by hand is far greater than the total of plants with mechanical stokers, but if the comparison is based on total horsepower developed, the figures show less difference." Hundreds of devices to assist in smoke prevention in hand fired plants have been patented, but as a very general rule, such devices are failures because they are not based on the fundamental principles of combustion as presented on pages 30 to 33. The steam jet, in some form or other, and applied at various points of the furnace, is by far the most common of these, and has for its best claim, that it mixes the air and combustible gases, but it must be used with better care and judgment than it ordinarily receives. Any gain that it may be able to effect is more than offset if it is allowed to run even a short time longer than necessary. Its services are only required at the moment of firing fresh fuel and for the short time that this fuel is giving off its volatile gases. Various devices make the operation of the jets *independent* of the fireman, the opening of the fire door turning on the steam, and a dashpot, suitably connected and adjusted, shuts off the jets after a proper interval.

The easiest and most nearly perfect solution of the problem of smoke prevention in any plant is a mechanical stoker properly set under the boiler. Boilers of wide difference in design have shown equal efficiency in steaming tests when using the same or similar furnaces, and it has come to be generally accepted that the difference in boiler design counts for less, efficiency alone being considered, than proper furnace design. The use of mechanical devices for firing coal reduces labor in the boiler room, but the *main object* of mechanical stoking is to feed a *steady, regulated* supply of *coal* and *air to* the furnace, permitting a *uniform evolution* of volatile gases so necessary to the effective solution of the smoke problem. They are usually designed to be self-cleaning, or to require a minimum of time and effort to clean, thus preventing admission of cold air through the boiler with its attendant objections.

Mechanical stokers may be classed as either overfeed or underfeed, the overfeed being in turn divided into front feed and side feed. The development of the successful stokers has been

along such lines as would facilitate the regulation of fuel and air and the removal of ashes.

One of the typical front feed stokers is the chain grate, illustrated in Fig. 5. It consists essentially of series of endless chain elements, side by side and of such number as to give desired width, these "chains" running over drums set several

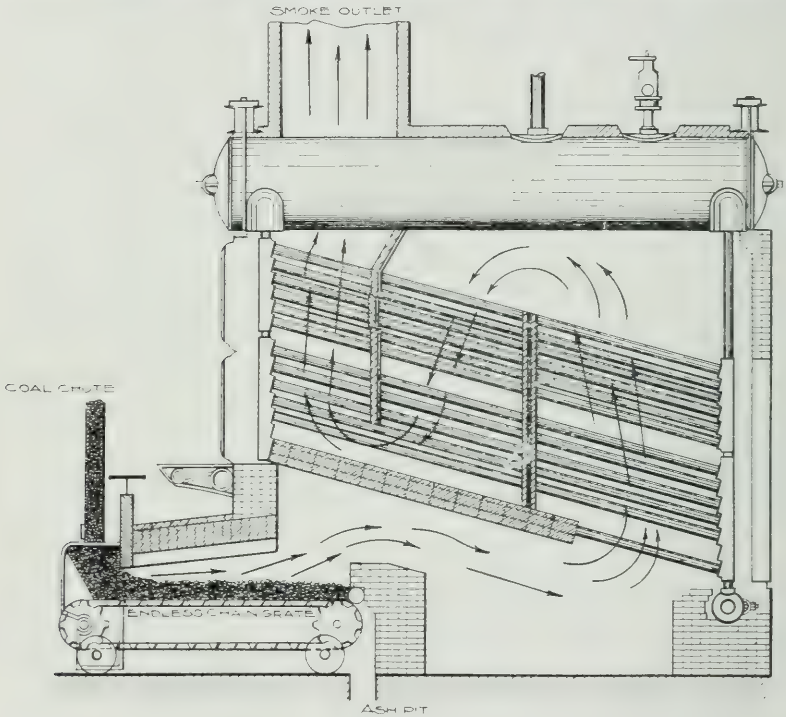


Fig. 5

Water Tube Boiler Equipped With Chain Grate Stoker.

feet apart and in turn supported on a frame work. This frame work is in turn mounted on flanged rollers, and can be drawn from the boiler setting, giving convenient access for inspection and repair. Coal is fed from a hopper which extends the full width of the grate, and as the chain moves into the furnace, driven by a ratchet device on the front drum, a uniform depth of coal, regulated by the position of a vertical plate on the back

side of the hopper, is drawn onto the grate and into the furnace. The ratchet takes its power from a constant speed motor or line shaft, and by proper setting of "notches", can feed a greater or less amount of coal into and through the furnace so that it will be completely burned by the time it gets to the back end and the ashes are dropped off over the back drum to a convenient storage pit or conveyor. These stokers are particularly adapted to the small sizes of non-caking coals, and are readily ad-

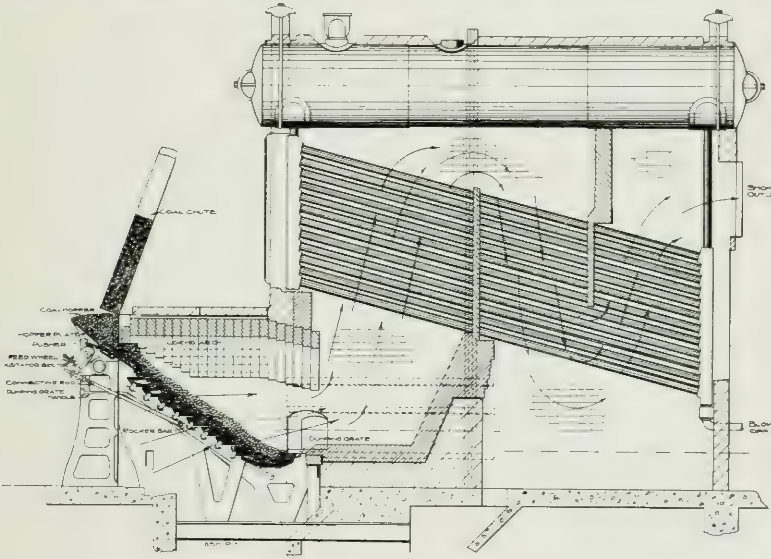


FIG. 6

WATER TUBE BOILER EQUIPPED WITH FRONT FEED INCLINED GRATE STOKER.

justed to sudden changes of load and to heavy overloads. They are open to the objection of considerable fine coal dropping through the grates, considerable excess air getting through the sides and back end of the grates, and the possibility of breaking of links due to "frozen" clinkers as the chain turns over the back sprocket or drum.

A typical front feed by means of an inclined and stepped grate, illustrated in Fig. 6, takes coal from a hopper extending the whole width of the grate, the coal being fed to the top

of the incline by a pusher plate which slides on the bottom of the hopper. The amount of coal fed depends on the length of stroke of the pusher plate and the number of such strokes per minute, these being controlled by adjustable bumper blocks and a variable speed stoker engine. This same engine also drives the eccentric from which is taken the motion to oscillate or reciprocate the steps of the grate in such a way as to gradually work the coal from in front of the pusher plate to the bottom of the incline, the extent and speed of these oscillations being so regulated that the coal will just be burned out by the time it gets to the bottom, where the ash can be dumped by manipulating proper levers and plates when a sufficient accumulation makes this necessary.

In the side feed stoker, Fig. 7, the grates slope from both sides and downward to the center at an angle slightly flatter than the angle of repose of the coal. The grate is made up of bars extending the full length of the slope, alternate bars being fixed and the bars in between so actuated as to raise the fuel from the surface of the fixed grate, set it down a little further on the slope, drop away from the fuel, and back up to its first position to repeat the process. Coal is taken from hoppers or magazines along the sides and top of the furnaces, fed onto the incline by pusher plates, and carried down at such a rate as to allow it to completely burn before it comes to the rotating clinker crushers located at the center and between the bottom ends of the two inclines. As in the preceding types, the rate of feeding of fuel may be regulated by changing the speed of the stoker engine, the steam from which is usually exhausted under the grate and assists in reducing and breaking up the clinker formation.

All three of the above types require some sort of an arch of refractory material sprung over the stoker to assist in the mixing of the air and gases, and to maintain the mixture at a temperature high enough to burn and give time for complete combustion before the heat absorbing surfaces of the boiler are reached. In some cases, air is drawn over outside surfaces, thus reducing radiation losses and supplying heated air to the fuel. Steam jets or curtains are also used to assist in mixing

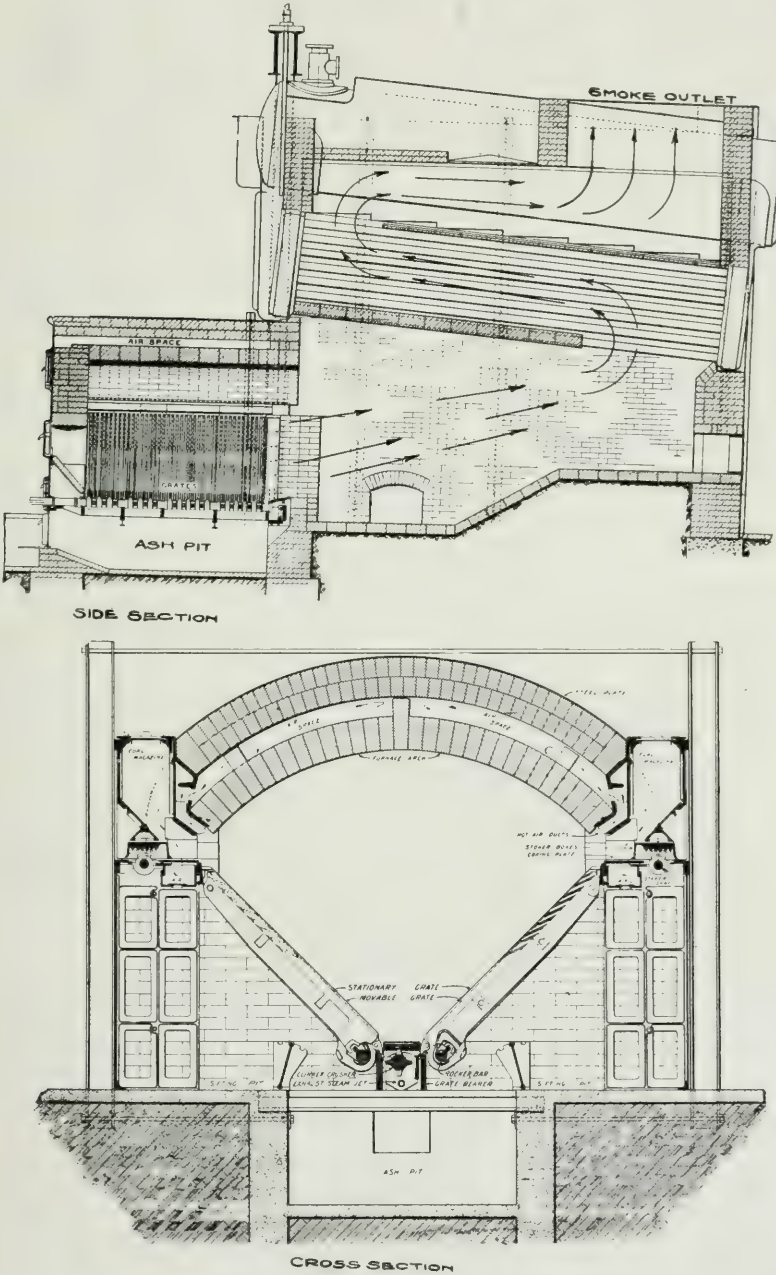


FIG. 7

WATER TUBE BOILER EQUIPPED WITH SIDE FEED INCLINED GRATE STOKER.

the air and gases and to direct the mixture against incandescent portions of the fuel bed.

Stokers of the underfeed class differ radically in design from the overfeed types. One of the underfeed types is shown in Fig. 8. The coal is fed from bunkers and downspouts, or shoveled by hand from a floor pile into a hopper, and falls down in front of a steam actuated ram which pushes the coal into

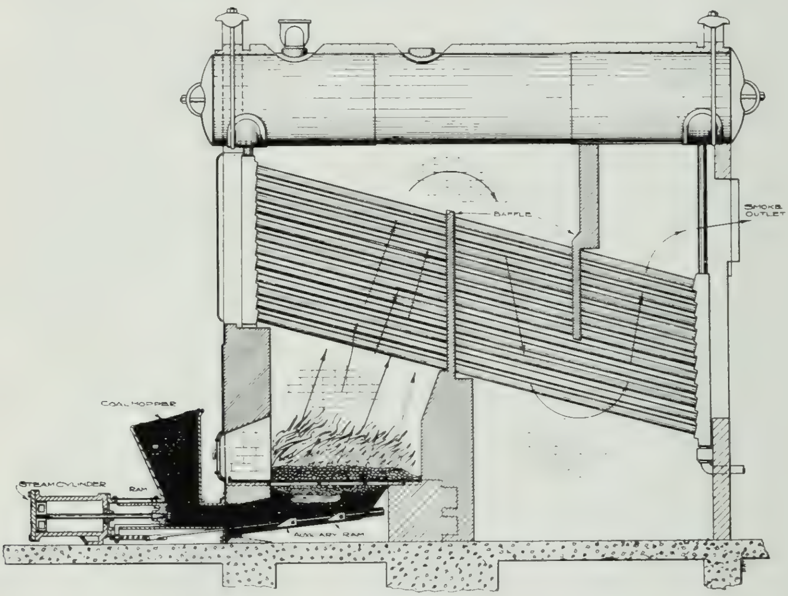


Fig. 8

Water Tube Boiler Equipped With Underfeed Stoker.

the bottom of a horizontal retort and under the burning fuel bed. This stoker requires a forced draft of considerable pressure furnished by a variable-speed, engine-driven fan. This air is admitted through renewable tuyere blocks along the upper edges of the retort at the level where the hydrocarbons are given off. The unburned refuse is usually fused to a clinker, which is gradually forced to the side of the heap of burning coal, and can be drawn through the doors at the front of the furnace. The valves to the steam actuated ram and the fan which sup-

plies the air are driven by an engine whose speed is varied automatically with the steam pressure. After a proper proportion of air and fuel has once been determined, this relation will be constant for all demands on the furnace, insuring complete combustion at all times, and an independence of atmospheric conditions which sometimes seriously affect the natural draft installations.

Another similar type feeds the coal into the retort with a screw instead of a ram, with essentially the same results. Still other types combine the design of rams and horizontal retorts with the inclined step grate as described under the overfeed stoker class and with very marked success. These are perhaps better adapted to very large installations. The disposal of ash and clinker is much more satisfactory than with the type described above. They require a rather heavy forced draft, are independent of atmospheric conditions, and respond quickly to changes of load and to heavy overloads.

The larger sizes of underfeed stokers, as used in power plants, have exactly the same advantages as specified for the smaller domestic furnaces, i. e., the hot fire is always on top; the radiation from the glowing fuel bed to the heat absorbing surfaces is not smothered by fresh fuel; and the volatile gases, passing through this fuel bed from below already mixed with sufficient air, burn completely and smokelessly within a short distance of the fuel bed. Hence the combustion space required over the fuel bed is less than with any other type, and a considerable saving in space and material in the setting is obtained.

It should be emphasized that stokers of any type are devices to *assist* in cutting down the waste due to smoke and incomplete combustion of volatile gases, and that they must be used with care and intelligence to get the best results. If they do not receive such attention, the investment and repairs are a needless expense, but with proper supervision, they may pay for themselves many times over by reducing the amount of coal used, making possible the use of a cheaper grade of coal, permitting large overloads on given equipment, and saving a considerable amount on labor costs.

THE SMOKE PROBLEM

Results of Smoky Conditions

It has already been explained that smoke is usually caused by the relatively cool surfaces of the boiler being too near the fuel bed, the flames being thus extinguished before complete combustion has had time to take place. As a result, solid carbon particles pass out of the stack in smoke. Contrary to general belief, the heating value of the carbon escaping as solid particles rarely represents more than 2 per cent of the heat units in the coal fired. If this were the only loss, the installation of expensive stokers and furnace settings would hardly be worth while. But fortunately black smoke is a signal of incomplete combustion with an accompanying loss many times greater than the loss of carbon particles alone. This loss, which results from the partial burning of carbon to carbon monoxide, and from the escape of unburned hydrogen and hydrocarbons, often amounts to 20 per cent of the heating value of the coal. The absence of smoke does not necessarily mean complete combustion. It may mean excessive dilution of air, which means excessive chimney losses. Many so-called "smoke consumers" operate in this way.

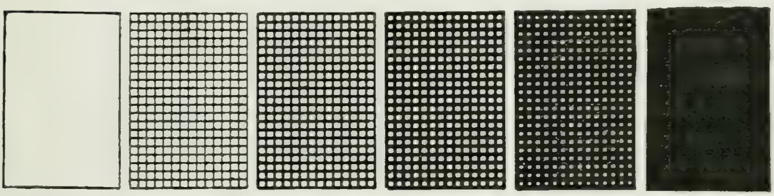
Smoking chimneys not only mean fuel wasted, but the damage resulting from the effects of the smoke is enormous and affects the public directly. According to a report of the Bureau of Mines, it has been estimated that in Cleveland, Ohio, this damage amounts to $12 per capita per annum. In Chicago, Illinois, the damage has been estimated to equal four-fifths of all the taxes levied in that city for municipal purposes, or a sum of at least 80 per cent of the cost of all the coal burned in that city. There are many ways in which this damage makes itself felt, the principal ones being: by increased expenditures for repairing and repainting exteriors and interiors of buildings, for artificial light made necessary by the decreased amount of sunlight, for laundering and cleaning, and by injury to vegetation.

Observation and Estimation of Smoke

The density of smoke is measured in many ways, but the most satisfactory at this time is by means of the Ringelmann Charts.[8] The plan requires six cards similar to those shown in Fig. 9, the cards in the illustration being very much reduced in size. The lines are so spaced as to give the effect of different percentages of blackness when placed at a distance of about 50 feet from the observer. The charts are numbered 0, 1, 2, 3, 4, and 5, and represent, respectively, 0, 20, 40, 60, 80, and 100 per cent of black smoke.

PER CENT OF DENSITY

0 20 40 60 80 100



0 1 2 3 4 5

Fig. 9

The Ringelmann Scale for Grading the Density of Smoke.

In making observations, the six cards are hung in line with the chimney at a point about 50 feet distant from the observer, at which distance the lines of the cards become invisible and the cards appear to be of different shades of gray, ranging from the very light gray to almost black. The observer glances alternately at the smoke and at the cards, makes observations continuously for one minute, and decides which card most nearly corresponds with the color of the smoke. The record is then made accordingly, noting the time. The color recorded is the estimated average density of the smoke during the entire minute, and records are made for each consecutive minute during the test. The average of all the records made during the test is taken as the average figure for the

[8] *Transactions* of the American Society of Mechanical Engineers, Vol. XXI, December, 1899.

smoke density during the test and the whole record is plotted on cross section paper to show the variations in density from time to time.

City Ordinances for Smoke Prevention

With the knowledge that smoke can be prevented, there has come an increasing demand from the public that steps be taken to prevent smoke being allowed to pollute the atmosphere. This demand has been expressed by the action of Chambers of Commerce and of various associations and leagues, and by the passage of smoke ordinances in most of the large cities of the country. As a result of the demands of the public, the ordinances of some of these cities require that all new plants be equipped properly and that old ones be remodeled, and permits are now necessary for the installation of all boilers and furnaces.

The following form of ordinance has been drafted by the Bureau of Mines[9] to suit the average conditions in cities of 50,000 to 200,000. While there are not many cities of this size in Wisconsin, this will give a general idea of the usual form of a smoke ordinance and of the requirements to be met.

Proposed Form of Smoke Ordinance for a Medium-Size City

An ordinance providing for smoke inspection and abatement in the city of ...., ...

The city council of the city of .... do ordain as follows:

Section 1. There is hereby created the office of smoke inspector, the compensation and duties connected therewith to be as hereinafter specified.

Section 2. The smoke inspector shall be appointed by the mayor, by and with the advice of the city council, and shall perform the duties of his office until removed from office or until his successor is appointed.

Section 3. The person so appointed shall be an engineer qualified by training and experience in the theory and practice of the construction of steam boilers and furnaces, also in the theory and practice of smoke abatement and prevention.

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[9] Flagg, S. B., *Smoke Abatement and City Smoke Ordinances*, (Bureau of Mines Bulletin No. 49).

Section 4. The salary of the smoke inspector shall be .... dollars ($..........) per annum.

Section 5. There shall be as many deputy smoke inspectors, assistant smoke inspectors, clerks, and stenographers as shall be provided by the city council; their compensation shall be fixed by the city council, and they shall be appointed by the smoke inspector as provided by law (or by the mayor, by and with the advice of the city council).

Section 6. The city council shall appoint a citizens' smoke-abatement committee composed of seven (7) representative members, who shall act as advisors to the mayor and to the smoke inspector upon matters pertaining to the organization or the conduct of the smoke-abatement work, or both. The smoke inspector shall at all times receive and place and keep on file all suggestions, recommendations, advice, or other communications that may be submitted to him in writing by the said committee.

Section 7. The citizens' smoke-abatement committee may procure the services of a consulting mechanical engineer of recognized ability who has had experience in the installation and operation of steam power and heating plants, and particularly in the prevention of smoke in such plants, to advise the smoke inspector and the committee upon engineering problems in connection with the smoke-abatement work whenever such advice is required; provided, however, that the total expense incurred for such consulting advice shall not exceed the sum allowed for such purpose by the city council.

Section 8. It shall be unlawful for the owner, lessee, or operator of any existing plant, or of any plant about to be constructed, for the production of power or heat, to proceed with the construction, reconstruction, or alteration of such plant until plans and specifications for such work shall have been submitted to the smoke inspector, approved by him, and a permit for the prosecution of such work issued. These plans and specifications shall show the nature and extent of the work to be done and the amount of power or heat to be supplied by such plant. Said plans and specifications shall also contain a statement of the kind of fuel to be used and shall show all provisions made for the purpose of obtaining complete combustion of the fuel to be used and for the purpose of preventing smoke. They shall also show that the space to be occupied by the plant and the location of the equipment therein will not prevent the proper operation of said equipment. Said plans and specifications shall also show that the room or apartment in which such plant is to be located is provided with doors, windows, or other means of ventilation sufficient to prevent the temperature in said room or apartment from rising to a point higher than 120° Fahrenheit, and sufficient also to provide that the air in said room or apartment may

be entirely renewed every ten minutes. Upon the approval of such plans, a duplicate copy shall be left with the smoke inspector, who shall then notify the building inspector, and it shall be the duty of said building inspector to see that the work is done in accordance with the plans and specifications. The foregoing provisions of this section shall not apply to the steam plants in buildings used exclusively for private residence purposes in which the number of families occupying apartments shall be less than five, nor to minor necessary or emergency alterations or repairs in any other plants, which alterations or repairs do not increase the capacity of said plants, or which do not involve any substantial alteration in structure, and which do not involve any alteration in the method or efficiency of smoke prevention.

Any person who shall violate this section shall be liable to a fine of twenty-five dollars ($25) for each day upon which he shall prosecute such alteration, change, or installation without a permit, and each day's violation shall constitute a separate offense.

Section 9. A fee of one dollar ($1) shall be charged for the inspection of plans and specifications for the erection, reconstruction, or alteration of any plant, this fee to include the issuing of a permit, in case such permit is granted.

Section 10. The emission of dense smoke within the city from the smokestack of any locomotive, steamboat, or steam tug for a period of more than seventy-five (75) seconds, except for a period or periods aggregating not to exceed 12 minutes in any one hour, during which period or periods the firebox is being cleaned or a new fire being built therein, is hereby declared a nuisance: *Provided*, that the fire engines or fire boats of the city fire department, or both of them, shall be exempt from these restrictions.

The emission of dense smoke within the city from the smokestack of any steam roller, steam derrick, steam pile driver, tar kettle, or other similar machine or contrivance, or from the smokestack or chimney of any building or premises, except for a period or periods aggregating not to exceed nine minutes in any one hour, during which period or periods the fire box is being cleaned or a new fire is being built therein, is hereby declared a nuisance.

Any nuisance such as the above specified may be summarily abated by the smoke inspector, or by anyone whom he may duly authorize for the purpose and such abatement may be in addition to the fine hereinafter provided.

Any person or persons, or corporation owning, operating, or in charge or control of any locomotive, steamboat, steam roller, steam derrick, steam pile driver, tar kettle, or other similar machine or contrivance, or of any building or premises who shall cause or permit the emission of dense smoke within the city, in contravention of the

provisions of this section, from the smokestack of any such locomotive, steamboat, steam tug, steam roller, steam pile driver, steam derrick, tar kettle, or other similar machine or contrivance, or from the smokestack or chimney or any building or premises so owned, controlled, or in charge of him, her, or them, shall be deemed guilty of a violation of this ordinance, and upon conviction thereof shall be fined not less than ten dollars ($10) nor more than one hundred dollars ($100) for each offense; and each day of such emission of dense smoke shall constitute a separate offense.

For the purpose of grading the density of smoke the Ringelmann smoke chart, as published and used by the Federal Bureau of Mines, shall be the standard of comparison. Smoke shall be considered "dense" when it is of greater density than No. 3 of the chart.

The provisions of this section shall not apply to detached private residences, nor to buildings used exclusively for private residence purposes, in which the number of families occupying apartments shall be less than five.

Section 11. Prosecutions for all violations of this ordinance shall be instituted by the smoke inspector and shall be prosecuted in the name of the city of ....

The issuance and the delivery by the smoke inspector of any permit for the construction, reconstruction, alteration, or repair of any plant or chimney connected with a plant, shall not be held to exempt any person or corporation to whom such permit has been issued or delivered, or who is in possession of any such permit, from prosecution on account of the emission or issuance of dense smoke caused or permitted by such person or corporation.

Section 12. The city shall provide such instruments, books, papers, and equipment as shall be necessary for the proper prosecution of the smoke-abatement work. The smoke inspector shall have charge of such instruments, books, papers, and equipment, and shall deliver the same to his successor in office.

Section 13. The smoke inspector shall cause to be kept in his office a complete record of all plans submitted and of all permits issued. He shall also cause to be kept a record of all stacks observed and of the smoke observations from any stack that is found to exceed the allowable time limit for dense smoke.

Section 14. The smoke inspector shall make a report of his work or the work done under his direction to the mayor and city council annually, on or before ...., and at other times as often as required by the city council.

Section 15. If any person acting on behalf of the city under the provisions of this ordinance shall take or receive any money or any valuable thing for the purpose of favoring any person or persons, he shall be fined.......... dollars ($.....) for each offense, and shall be dis-

missed from the service; or if the inspector shall issue any permit as mentioned in section 8 of this ordinance, without thoroughly examining the plans and specifications for the work for which the permit is issued, he shall be fined ...... dollars ($....) for each offense.

The Problem in a Small City

The problem in the city of, say, 20,000 population, will usually require a different method of attack than has been outlined for the larger city. A small city has comparatively few new boiler plants, reconstructions, or alterations in the course of a year, so that the policing phase becomes of major importance. However, the small city ought to protect itself and its manufacturers from the installation of improperly designed furnaces and thereby save heavy expenditures for reconstruction in later years. The board of trade or a similar civic organization can authorize a committee to obtain information concerning furnace designs or methods of operation developed in other cities, or to inform themselves so that they may be able to direct those interested to competent advisors on the subject of smoke prevention. With a strong public sentiment to supplement its efforts, such a civic organization should be able to accomplish much good in the smaller cities that can afford no organized municipal effort toward smoke abatement.

PUBLICATIONS ON THE BURNING OF SOFT COAL

The following bulletins are excellent papers on the soft coal problem, and may be obtained for the asking or at a very small expense.

Kreisinger, H., *Hand Firing Soft Coal under Power Plant Boilers* (Technical Paper No. 80 of the Bureau of Mines).

Randall, D. T., and Weeks, H. W., *The Smokeless Combustion of Coal in Boiler Furnaces* (Bulletin No. 40 of the Bureau of Mines).

Flagg, S. B., *Smoke Abatement and Smoke Ordinances* (Bulletin No. 49 of the Bureau of Mines).

Breckenridge, L. P., and Flagg, S. B., *Saving Fuel in Heating a House* (Technical Paper No. 97 of the Bureau of Mines).

These bulletins can be obtained by writing to the Director of the Bureau of Mines, Washington, D. C.

Breckenridge, L. P., *How to Burn Illinois Coal Without Smoke* (Bulletin No. 15, University of Illinois Experiment Station).

The Economical Purchase and Use of Coal for Heating Homes, with Special Reference to Conditions in Illinois (Circular No. 4, University of Illinois Experiment Station, Price ten cents).

These two bulletins can be obtained by writing to the University of Illinois Experiment Station, Urbana, Illinois.

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(*Complete in ten numbers, with title-page, table of contents, and index.*)

No. 1. A complete test of modern American transformers of moderate capacities, by Arthur Hillyer Ford. 1896. 88 p. 35 cents.

No. 2. A comparative test of steam injectors, by George Henry Trautmann. 1897. 34 p. 25 cents.

No. 3. The superintendent of bridges and buildings, by Onward Bates. 1898. 30 p. 25 cents.

No. 4. Some unrecognized functions of our state universities, by John Butler Johnson. 1899. 20 p. 25 cents. *Out of print.*

No. 5. The transcontinental triangulation along the thirty-ninth parallel, by John Fillmore Hayford. 1900. 22 p. 6 pl. 25 cents.

No. 6. The chemical engineer, by Magnus Swenson. 1900. 13 p. 25 cents.

No. 7. Recently improved methods of sewage disposal, by John Butler Johnson. 1900. 19 p. 25 cents.

No. 8. An experimental study of the corrosion of iron under different conditions, by Carl Hambuechen. 1900. 47 p. 19 pl. 30 cents.

No. 9. The progress of the ceramic industry, by Edward Orton, Jr. 1903. 23 p. 25 cents.

No. 10. The effect of frequency on the steadiness of light emitted from an incandescent lamp, by Harold Seaman. 1903. 47 p. 25 cents.

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(*Complete in six numbers, with title-page and table of contents.*)

No. 1. An investigation of rotations produced by current from a single-phase alternator, by Arthur Curtis Scott. 1904. 162 p. 50 cents.

No. 2. The sources of water supply in Wisconsin, by William Gray Kirchoffer. 1905. 88 p. 50 cents.

No. 3. An investigation of the borides and the silicides, by Oliver Patterson Watts. 1906. 68 p. 30 cents.

No. 4. Tests on reinforced cencrete beams, by Ernest Anthony Moritz. 1907. 76 p. 30 cents.

No. 5. A comparison of the effects of frequency on the light of incandescent and Nernst lamps, by Frederick William Huels. 1907. 51 p. 25 cents.

No. 6. Investigation of centrifugal pumps: Part I, A discussion of the theory of the centrifugal pump and tests of a six inch vertical centrifugal pump, by Clinton Brown Stewart. 1907. 141 p. 50 cents.

VOLUME IV

(*Complete in six numbers, with title-page and table of contents.*)

No. 1. Tests on plain and reinforced concrete, series of 1906, by Morton Owen Withey. 1907. 66 p. 25 cents.

No. 2. Tests on plain and reinforced concrete, series of 1907, by Morton Owen Withey. 1907. 69 p. 25 cents.

No. 3. An investigation of the hydraulic ram, by Leroy Francis Harza. 1908. 110 p. 25 cents.

No. 4. Investigation of flow through large submerged orifices and tubes, by Clinton Brown Stewart. 1908. 83 p. 25 cents.

No. 5. Current practice in steam engine design, by Ole N. Trooien. 1908. 79 p. 25 cents.

No. 6. Self-excited asynchronous polyphase generators, by Lewis Fussell. 1909. 87 p. 25 cents.

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(*Complete in six numbers, with title-page and table of contents.*)

No. 1. Investigation of railway train lighting, by Edward Wray. 1908. 142 p. 50 cents.

No. 2. Tests on plain and reinforced concrete columns, by Morton Owen Withey. 1909. 45 p. 25 cents.

No. 3. Investigation of centrifugal pumps: Part II, Experiments with a six inch horizontal centrifugal pump comparing the efficiency of circular and spiral cases and showing the effect of air leakage, by Clinton Brown Stewart. 1909. 129 p. 50 cents.

No. 4. Sewage purification with special reference to Wisconsin conditions, by George Jacob Davis, and James Ten Broeck Bowles. 1909. 88 p. 50 cents.

No. 5. Tests on bonds between concrete and steel in reinforced concrete beams, by Morton Owen Withey. 1909. 64 p. 25 cents.

No. 6. Relation of experiments to the theory of the tangential water wheel, by Daniel Webster Mead. 1909. 104 p. 40 cents.

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No. 3. Long distance transmission of steam and its effect on power plant economics, by Halsten Joseph Thorkelson. 1910. 34 p. 25 cents.

No. 4. Investigation of hydraulic curve resistance, by George Jacob Davis. 1911. 60 p. 25 cents.

No. 5. The flow of streams and the factors that modify it, with special reference to Wisconsin conditions, by Daniel Webster Mead. 1911. 191 p. 40 cents.

No. 6. Some alloys of calcium, by James Miller Breckenridge. 1911. 40 p. 25 cents.

No. 7. An investigation of the air lift pump, by Geo. Jacob Davis, Jr., and Carl Robert Weidner. 1911. 168 p. 40 cents.

VOLUME VII

(Complete in seven numbers, with title-page and table of contents.)

No. 1. Tests on reinforced concrete columns, series of 1910, by Morton Owen Withey. 1911. 116 p. 40 cents.

No. 2. Theory and test of an overshot water wheel, by Carl Robert Weidner. 1913. 136 p. 40 cents.

No. 3. Investigation of hydraulic curve resistance, experiments with three inch pipe, by Leland Rella Balch. 1913. 51 p. 25 cents.

No. 4. Test of a jet pump, by Leland Rella Balch. 1913. 16 p. 25 cents.

No. 5. A test of an eight foot flash wheel, by Leland Rella Balch. 1913. 51 p. 25 cents.

No. 6. Investigation of flow through large submerged orifices: Part III, Experiments with submerged draft tubes, by George Jacob Davis, Jr., and Leland Rella Balch. 1914. 58 p. 25 cents.

No. 7. Tar forming temperatures of American coals, by Otto Carter Berry. 1914. 74 p. 25 cents.

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No. 3. Investigation of flow through four inch submerged orifices and tubes, by Leland Rella Balch. 1914. 32 p. 25 cents.

No. 4. High versus low antennae in radio telegraphy and telephony, by Edward Bennett. 1916. 68 p. 25 cents.

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